HOMOLOGICAL ALGEBRA AND PROBLEMS IN COMBINATORICS AND GEOMETRY

A Dissertation

by

ŞTEFAN OVIDIU TOHĂNEANU

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2007

Major Subject: Mathematics

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Approved by:

Chair of Committee,	Henry Schenck
Committee Members,	Paulo Lima-Filho
	Peter Stiller
	John Keyser
Head of Department	Albert Boggess

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ABSTRACT

Homological Algebra and Problems in Combinatorics and Geometry. (May 2007)

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This dissertation uses methods from homological algebra and computational commutative algebra to study four problems. We use Hilbert function computations and classical homology theory and combinatorics to answer questions with a more applied mathematics content: splines approximation, hyperplane arrangements, configuration spaces and coding theory.

In Chapter II we study a problem in approximation theory. Alfeld and Schumaker give a formula for the dimension of the space of piecewise polynomial functions (splines) of degree d and smoothness r. Schenck and Stiller conjectured that this formula holds for all $d \geq 2r+1$. In this chapter we show that there exists a simplicial complex Δ such that for any r, the dimension of the spline space in degree $d = 2r$ is not given by this formula.

Chapter III is dedicated to formal hyperplane arrangements. This notion was introduced by Falk and Randell and generalized to $k-$formality by Brandt and Terao. In this chapter we prove a criteria for $k-$formal arrangements, using a complex constructed from vector spaces introduced by Brandt and Terao. As an application, we give a simple description of $k-$formality of graphic arrangements in terms of the homology of the flag complex of the graph.

Chapter IV approaches the problem of studying configuration of smooth rational curves in $\mathbb{P}^2$. Since an irreducible conic in $\mathbb{P}^2$ is a $\mathbb{P}^1$ (so a line) it is natural to

ask if classical results about line arrangements in $\mathbb{P}^2$, such as addition-deletion type theorem, Yoshinaga criterion or Terao's conjecture verify for such configurations. In this chapter we answer these questions. The addition-deletion theorem that we find takes in consideration the fine local geometry of singularities. The results of this chapter are joint work with H. Schenck.

In Chapter V we study a problem in algebraic coding theory. Gold, Little and Schenck find a lower bound for the minimal distance of a complete intersection evaluation codes. Since complete intersections are Gorenstein, we show a similar bound for the minimal distance depending on the socle degree of the reduced zero-dimensional Gorenstein scheme. The results of this chapter are a work in progress.

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I would also like to thank the members of my committee: John Keyser, Paulo Lima-Filho and Peter Stiller. Along the way they gave me very helpful advice that also helped me with the final touches of my dissertation. I am very grateful to all the professors in the Math Department at Texas A&M University. In particular, thank you to Hal Schenck, Peter Stiller, Paulo Lima-Filho, Joseph Landsberg and Frank Sottile. Their classes or seminars about algebraic geometry, commutative algebra or differential geometry were very helpful to me in building a good mathematical background. I would like to thank our Head of Department, Al Boggess, and all the auxiliary staff, for the perfect environment they created in this department and for all the help I received along the way. Also I would like to thank my colleague Terry McDonald, for providing me the template used to type this dissertation.

In the end, I thank my wife Mari. I don't have words to express how grateful I am to her for all the support and encouragement she gave me. Her kindness, calm and joy are the breath of fresh air I always need. I thank my parents and my brother for all the support they gave me from far away in my home country. I dedicate this dissertation to all of them.

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CHAPTER I

INTRODUCTION

A. Some Homological Algebra

Definition I.1. A sequence of modules $\{M_i\}_{i\geq 0}$ over the same ring R and homomorphisms between them $d_i : M_i \longrightarrow M_{i-1}$ is called a *complex* if $d_i \circ d_{i+1} = 0, \forall i$. A complex is usually represented as follows:

$$(M_*, d_*) : \cdots \longrightarrow M_i \xrightarrow{d_i} M_{i-1} \longrightarrow \cdots \xrightarrow{d_1} M_0 \xrightarrow{d_0} 0.$$

At each step i define the *homology* of the complex to be the $R-$modules $H_i(M_*, d_*) = ker(d_i)/im(d_{i+1})$.

Definition I.2. An $R-$ module M is called *free of rank* k if it is isomorphic as $R-$modules to R^k. In other words, a free module is the natural generalization of a vector space: M is generated by k linearly independent elements.

A complex is *exact* if all the homology modules are 0. A particular case of exact complexes are the free resolutions.

Definition I.3. Let M be an $R-$module. *A free resolution* of M is an exact complex as above with $M_0 = M$ and $\forall i \geq 1$, M_i are free $R-$modules.

Definition I.4. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and $I \subset R$ an ideal. R/I is *Cohen-Macaulay* if and only if R/I as an $R-$module has a minimal free resolution of length $codim(I)$.

This dissertation follows the style of Advances in Computational Mathematics.

Note that in general the length is $\geq codim(I)$ and if I is homogeneous ideal, Hilbert's syzygy theorem says that the length of a minimal free resolution is $\leq n$.

Often it is difficult to find a free resolution for an arbitrary module. *The Hilbert function* is the invariant closest to the free resolution that can give us informations about the module we study.

Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be a graded ring. An $R-$ module M is called *graded* if $M = \bigoplus_{i\in\mathbb{Z}} M_i$ and for any $m \in M_d$ and $r \in R_e$, we have $r \cdot m \in M_{d+e}$.

Definition I.5. If M is a finitely generated graded module over $R = \mathbb{K}[x_1, \ldots, x_n]$, with grading given by the degree, the numerical function

$$HF(M, d) = \dim_{\mathbb{K}} M_d$$

is called *the Hilbert function* of M.

For a degree i sufficiently large, the Hilbert function becomes a polynomial called *the Hilbert polynomial*. This polynomial allows the computation of two very important invariants of a projective variety $X \subseteq \mathbb{P}^n$:

- *dimension*: the number of times it is possible to intersect X with generic hyperplanes without the resulting intersection being empty.
- *degree*: the number of points in the zero-dimensional object obtained after the final intersection above (which is nonempty).

For a homogeneous ideal $I \subseteq \mathbb{K}[x_0, \ldots, x_n]$ the Hilbert polynomial can be written:

$$HP(R/I, i) = \frac{a_m}{m!} i^m + \frac{a_{m-1}}{(m-1)!} i^{m-1} + \cdots .$$

The dimension of $V(I) \subseteq \mathbb{P}^n$ is m and the degree is a_m.

The following proposition shows how to compute the Hilbert function from the free resolution.

Proposition I.1. *The following are true:*

1. *If* $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ *is an exact sequence of graded* $R-$*modules, then* $HF(N, i) = HF(M, i) + HF(P, i)$.
2. $HF(\mathbb{K}[x_0, \ldots, x_n], i) = \binom{n+i}{i}$.

Example I.1. Let $X = \{(1:0:0), (0:1:0), (0:0:1), (1:1:1)\} \subset \mathbb{P}^2$ be a set of four points. Let $I_X = \langle (x-y)z, x(y-z) \rangle$ be the ideal of X. Since X is a zero-dimensional variety consisting of 4 points, then $HP(R/I_X, i) = 4$.

Let's check this by using the proposition above. The free resolution of R/I_X is:

$$0 \longrightarrow R(-4) \xrightarrow{\left[\begin{smallmatrix} -g \\ f \end{smallmatrix}\right]} R^2(-2) \xrightarrow{[\,f\ g\,]} R \longrightarrow R/I_X \longrightarrow 0,$$

where f and g are the two generators of I_X.

For $i = 0$, the basis for $(R/I_X)_0 = k$ is $\{1\}$, so $HF(R/I_X, 0) = 1$.

For $i = 1$, the basis for $(R/I_X)_1$ is $\{x, y, z\}$ and so $HF(R/I_X, 1) = 3$.

For $i = 2$, the basis for R_2 is $\{x^2, y^2, z^2, xy, xz, yz\}$. In $(R/I_X)_2$, these elements satisfy 2 relations given by the 2 generators of I_X, therefore $HF(R/I_X, 2) = 4$.

In general

$$HF(R/I_X, i) = \binom{i+2}{2} - 2\binom{i}{2} + \binom{i-2}{2},$$

which for $i \geq 2$ is equal to 4, the Hilbert polynomial of R/I_X.

B. Some Combinatorics and Simplicial Complexes

Definition I.6. An abstract simplicial complex Δ on a vertex set V is a set of subsets of V such that

- If $v \in V$, then $\{v\} \in \Delta$.
- If $\tau \in \Delta$ and $\sigma \subset \tau$, then $\sigma \in \Delta$.

A set in Δ with $i+1$ elements is called an $i-$*face* of Δ. The *dimension* of Δ is the dimension of the largest face (*simplex*) it contains.

Example I.2. Let $V = \{a, b, c, d\}$ and let $\Delta = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{c, a\}, \{c, b\}, \{a, b\}, \{a, d\}, \{b, d\}, \{c, a, b\}\}$. It is easy to check that this is a simplicial complex of dimension 2.

For a simplicial complex Δ fix a ring R and define free $R-$ modules C_i in the following way: the generators of C_i are the oriented (e.g. pick and ordering of the vertices) $i-$simplices of Δ and the relations are given by the permutations (modulo signs of the permutation) of the elements in each simplex. For C_2 in the example above, $[c, a, b]$ is the same as $-[a, c, b]$, etc.

For each i define a map (*the boundary map*) $C_i \xrightarrow{\partial} C_{i-1}$, given by

$$\partial([v_{j_0}, \ldots, v_{j_i}]) = \sum_{k=0}^{i} (-1)^k [v_{j_0}, \ldots, \hat{v_{j_k}}, \ldots, v_{j_i}].$$

Each C_i has rank equal to the number of $i-$faces of Δ, and together with the boundary map they form a chain complex (easy to check that $\partial\partial = 0$). The homology of this complex is denoted with $H_i(\Delta)$. In most of the cases these complexes are not exact. For example, $rank H_0(\Delta) =$ the number of connected components of Δ. The rank of $H_i(\Delta)$ is called *the* i^{th} *Betti number of* Δ.

Under certain conditions, any topological space can be "approximated" to a simplicial complex, and many topological invariants can be computed using these objects and their homology (cohomology). Also there are very nice connections between the study of simplicial complexes and the commutative algebra of the Stanley-Reisner ring.

Example I.3. Computation of the homology for the simplicial complex in Example I.2.

C_0 has basis $\{[a],[b],[c],[d]\}$.

We can pick basis for C_1 to be $\{[c,a],[c,b],[a,b],[a,d],[b,d]\}$.

We can pick basis for C_2 to be $\{[c,a,b]\}$.

The chain complex for Δ is just

$$0 \longrightarrow R \xrightarrow{\partial_2} R^5 \xrightarrow{\partial_1} R^4 \longrightarrow 0.$$

The matrices of the boundary maps, in the given basis are:

$$\partial_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \partial_1 = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Since Δ is connected, $\dim H_0(\Delta) = 1$. So $rank(\partial_1) = 4 - 1 = 3$. Thus $dim(\ker \partial_1) = 5 - 3 = 2$.

Obvious, $\ker \partial_2 = 0$. So $H_2(\Delta) = 0 - 0 = 0$ and $rank(\partial_2) = 1 - 0 = 1$. Thus, $\dim H_1(\Delta) = 2 - 1 = 1$.

Example I.4. The following example will appear later in the dissertation.

Let G be a connected graph with no loops or multiple edges. From G we can build a simplicial complex, called *the flag (clique) complex*, as follows:

- 0-faces = the vertices of G;
- 1-faces = the edges of G;

- for $i \geq 2$, $i-$faces $=$ the K_{i+1} subgraphs of G (i.e. the complete subgraphs of G on $i+1$ vertices).

The homology of the chain complex of the flag complex associated to a graph will yeld us a useful criteria related to a problem in hyperplane arrangements.

C. Hyperplane Arrangements

Definition I.7. Let $\mathbb{K}$ be a field and let V be a $\mathbb{K}-$vector space of dimension ℓ. A hyperplane H in V is an affine subspace of dimension $\ell-1$. *A hyperplane arrangement* is a finite set of hyperplanes in V

A hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ is said to be *central* if all the H_i's pass through the origin of V. In this case each hyperplane is defined by the vanishing of a linear form in the variables $\{x_1, \dots, x_\ell\}$, the dual basis in V^*. An arrangement is *essential* if rank $r(\mathcal{A}) = codim(\cap_{i=1}^n H_i)$ is equal to ℓ.

Example I.5. Let G be a connected graph with no loops or multiple edges. The graphic arrangement associated to G is

$$\mathcal{A}_G = \{\ker(\alpha_{ij}) | \alpha_{ij} = x_i - x_j, i < j \text{ and } [ij] \text{ is an edge in } G\}.$$

A graphic arrangement is central, but is not essential since the subspace spanned by $(1, 1, \dots, 1)$ is contained in all the hyperplanes.

As noted in the previous paragraph, each $H \in \mathcal{A}$ is the kernel of a polynomial, α_H, of degree 1 (linear form for the case of central arrangements) in $S = \mathbb{K}[x_1, \dots, x_\ell]$. The product

$$Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

is called *the defining polynomial* of $\mathcal{A}$.

Definition I.8. For a central hyperplane arrangement $\mathcal{A} = \cup_{i=1}^{n} H_i$ in V, where $H_i = \ker(\alpha_i)$, α_i is a linear form in V^*, define the following module of derivations:

$$D(\mathcal{A}) = \{\theta \in Der_{\mathbb{K}}(S) | \theta(\alpha_i) \in \alpha_i S, \forall i = 1, \dots, n\}$$

where $S = Sym(V^*)$ and $Der_{\mathbb{K}}(S) = \{\theta : S \longrightarrow S, \mathbb{K}-\text{linear} | \theta(fg) = f\theta(g) + g\theta(f)\}$. We say that $\mathcal{A}$ is a *free arrangement* iff $D(\mathcal{A})$ is a free $S-$module.

$D(\mathcal{A})$ inherits the grading from S. For a free hyperplane arrangement $\mathcal{A}$, the degrees of the elements in the basis are called *exponents*.

Proposition I.2. *([17]) The following are true for any hyperplane arrangement:*

1. $D(\mathcal{A}) = D_0(\mathcal{A}) \oplus < E >$ *where* $E = \sum_{i=1}^{\ell} x_i \frac{\partial}{\partial x_i}$ *is the Euler derivation.*
2. $D(\mathcal{A}) = \{\theta \in Der_{\mathbb{K}}(S) | \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S\}$.
3. *Every* $\theta \in D_0(\mathcal{A})$ *corresponds naturally to a syzygy on the generators of the Jacobian ideal,* $J = \langle \frac{\partial Q}{\partial x_1}, \dots, \frac{\partial Q}{\partial x_\ell} \rangle$.

The fundamental test to check if an arrangement is free or not is *Saito's Criteria*; after giving the statement, we'll show how it transforms into a question in commutative algebra.

Theorem I.1. *([20]) The following are equivalent:*

1. $D(\mathcal{A})$ *is free with basis* $\theta_i = \sum_{j=1}^{\ell} a_{ij} \frac{\partial}{\partial x_i}, i = 1, \dots, \ell,$
2. $\langle \det(a_{ji}) \rangle = \langle Q(\mathcal{A}) \rangle$.

We may always assume θ_1 to be the Euler derivation. Expanding the determinant with respect to the first column $(x_1, \dots, x_\ell)^\tau$, and using the Euler formula for the homogeneous polynomial $Q(\mathcal{A})$, shows that the $(n-1) \times (n-1)$ minors of the matrix

$(\theta_2|\theta_3|\cdots|\theta_\ell)$ must be the generators of the Jacobian ideal of $Q(\mathcal{A})$. By the Hilbert-Burch theorem, this fact is equivalent to S/J being Cohen-Macaulay. Note that $codim(J) = 2$, since a prime minimal over J is generated by 2 linear forms.

Throughout this dissertation, we will work over the field of complex numbers, so $V \simeq \mathbb{C}^\ell$. An arrangement seems like a simple object, but in fact the topology of $V \setminus \mathcal{A}$ is quite complicated. One advantage of a free arrangement is that due to Terao ([30]) we can compute the Poincare polynomial of the complement, and more important, it factors completely.

Theorem I.2. *([30]) Let $\mathcal{A}$ be a free arrangement with exponents $\{a_1, \ldots, a_\ell\}$. Then the Poincare polynomial of the complement factors:*

$$\pi(\mathbb{C}^\ell \setminus \mathcal{A}, t) = \prod_{i=1}^{\ell}(1 + a_i t).$$

Example I.6. Consider the braid arrangement (see Figure 1 below) $A_3 \subseteq \mathbb{P}^2$ with defining polynomial

$$Q(\mathcal{A}) = xyz(x-y)(x-z)(y-z).$$

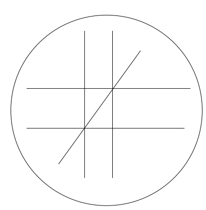

Fig. 1. A_3 arrangement.

R/J has free resolution:

$$0 \longrightarrow R(-8) \oplus R(-7) \longrightarrow R^3(-5) \longrightarrow R \longrightarrow R/J \longrightarrow 0$$

so R/J is Cohen-Macaulay, and hence $\mathcal{A}$ is free with exponents $\{1, 2, 3\}$.

CHAPTER II

SMOOTH PLANAR R-SPLINES OF DEGREE $2R$

A. Introduction

Let Δ be a connected finite simplicial complex which is supported on $|\Delta| \subset \mathbb{R}^2$. That is Δ is a triangulation of a region in the plane. Let $r \geq 0$ be an integer and let $\hat{\Delta}$ be the projective cone of Δ with origin in $\mathbb{R}^3$. Let $R = \mathbb{R}[x, y, z]$ and define the space of $r-$splines of degree k to be:

Definition II.1. $C^r_k(\hat{\Delta}) = \{F : |\hat{\Delta}\ | \ \rightarrow \mathbb{R} | F$ is continuously differentiable of order r and $F|_{\hat{\sigma}} \in R_k$ for all triangles $\sigma \in \Delta\}$.

Splines are defined by the following local condition: if σ_1, σ_2 are two triangles of Δ that share an edge defined by the equation $L = 0$, then $F \in C^r_k$ iff $F|_{\hat{\sigma_1}} - F|_{\hat{\sigma_2}} \in \langle L^{r+1} \rangle_k$.

There has been a tremendous amount of work to find the dimension of C^r_k. In [1], Alfeld and Schumaker give a formula for this dimension, if the degree k is $\geq 3r + 1$. The formula was conjectured to be true for $k \geq 2r + 1$, by Schenck and Stiller. In the next section we will show that this conjecture is tight. We first give an example:

Example II.1. Let's consider the following triangulation Δ and compute $\dim C^1_3(\Delta)$ (Figure 2 below).

The 3 vertices on the top horizontal have coordinates: $(0, 1), (1, 1)$ and $(2, 1)$.

The 3 vertices on the bottom horizontal have coordinates: $(0, 0), (1, 0)$ and $(2, 0)$.

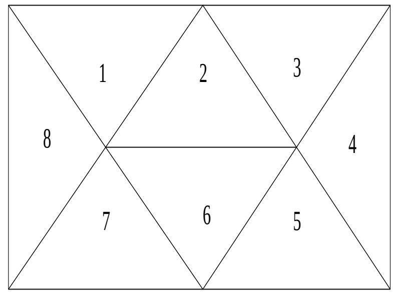

Fig. 2. Δ triangulation with $\dim C_3^1(\Delta) = 23$.

A spline of degree 3 and smoothness 1 on Δ corresponds to a vector $F = (f_1, \ldots, f_8)$, where f_i are homogeneous polynomials in R of degree 3, such that the following relations hold:

$$f_1 - f_2 = (x-y)^2 g_{12}$$

$$f_2 - f_3 = (x+y-4z)^2 g_{23}$$

$$f_3 - f_4 = (x-y-2z)^2 g_{34}$$

$$f_4 - f_5 = (x+y-4z)^2 g_{45}$$

$$f_5 - f_6 = (x-y-2z)^2 g_{56}$$

$$f_6 - f_7 = (x+y-2z)^2 g_{67}$$

$$f_7 - f_8 = (x-y)^2 g_{78}$$

$$f_8 - f_1 = (x+y-2z)^2 g_{18}$$

$$f_2 - f_6 = (y-z)^2 g_{26}$$

f_1 can be any polynomial of degree 3 and as long as we find the linear forms g_{ij}

subject to the following relations:

$$0 = (x+y-4z)^2(-g_{23}-g_{45}) + (x-y-2z)^2(-g_{34}-g_{56}) + (y-z)^2 g_{26}$$
$$0 = (x-y)^2(g_{12}+g_{78}) + ((x+y-2z)^2(g_{67}+g_{18}) + (y-z)^2 g_{26}$$

then F is determined.

These relations imply that the degree 1 syzygies (P_1, Q_1, g_{26}) on the generators of the ideal $I_1 = \langle (x+y-4z)^2, (x-y-2z)^2, (y-z)^2 \rangle$ and (P_2, Q_2, g_{26}) on the generators of the ideal $I_2 = \langle (x-y)^2, (x+y-2z)^2, (y-z)^2 \rangle$ determine F. This is equivalent to finding constants $a_1, b_1, a_2, b_2 \in \mathbb{R}$ such that

$$a_1(x-3z) + b_1(y-z) = a_2(x-z) + b_2(y-z) = g_{26}.$$

So we have 1-dimensional family of such syzygies. With this,

$$\dim C_3^1(\Delta) = \binom{5}{2} + 12 + 1 = 23.$$

The Alfeld-Schumaker formula for the case when $r = 1$ is:

$$\dim C_k^1 = \binom{k+2}{2} + e_0\binom{k}{2} - v_0(\binom{k+2}{2} - 3) + s,$$

where e_0 = number of interior edges, v_0 = number of interior vertices, s = number of interior vertices with just 2 different slopes of the edges incident to the interior vertex. So, for this example, $\dim C_3^1(\Delta) = 10 + 9 \cdot 3 - 2 \cdot 7 + 0 = 23$.

Using this example we will show in the next section that the conjecture of Schenck and Stiller is the best possible. Notice also that in this example we had to find some special syzygies on the generators of two ideals.

As shown in [28],

$$\dim C^r_d(\Delta) = \dim C^r_d(\hat{\Delta}) = L(\Delta, r, d) + \dim N_d$$

where $L(\Delta, r, d)$ is the Alfeld-Schumaker formula and N is a graded $R = \mathbb{R}[x, y, z]$ module of finite length. Lemma 3.8 of [28] contains the following description: N is the quotient of a free module generated by the totally interior edges (those edges with no vertex $\subseteq \partial\Delta$), modulo the syzygies at each interior vertex. The generators of N are shifted so that they have degree $r + 1$. This description seems cumbersome, but as we'll see in the example below, it is fairly easy to work with.

In the terms above, the conjecture of [26] is that N vanishes in degree $2r+1$. Our goal is to show that this bound is the best possible, so we want to find a configuration Δ such that for all r, $N_{2r} \neq 0$.

Consider the triangulation Δ in the example above. To find N, we begin by determining the minimal free resolutions for the ideals $I_i = \mathcal{J}(v_i)$ for v_1 and v_2 the interior vertices. We have:

$$I_1 = \langle (x + y - 2z)^{r+1}, (x - y)^{r+1}, (y - z)^{r+1} \rangle$$

$$I_2 = \langle (x + y - 4z)^{r+1}, (x - y - 2z)^{r+1}, (y - z)^{r+1} \rangle.$$

These ideals are in $R = \mathbb{R}[x, y, z]$. Notice that $y - z$ is the linear form vanishing on the totally interior edge. With the change of variables given by the matrix

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & 2 \\ 1 & 1 & -4 \end{bmatrix}$$

we can suppose that

$$I_1 = \langle x^{r+1}, (x+y)^{r+1}, y^{r+1} \rangle, \text{ and } I_2 = \langle z^{r+1}, (z+y)^{r+1}, y^{r+1} \rangle.$$

The minimal free resolutions for these ideals are:

$$0 \longrightarrow R^2 \xrightarrow{\left[\begin{smallmatrix} A_1 & D_1 \\ B_1 & E_1 \\ C_1 & F_1 \end{smallmatrix}\right]} R^3 \xrightarrow{\left[\begin{smallmatrix} x^{r+1} & (x+y)^{r+1} & y^{r+1} \end{smallmatrix}\right]} R \longrightarrow R/I_1 \longrightarrow 0$$

and

$$0 \longrightarrow R^2 \xrightarrow{\left[\begin{smallmatrix} A_2 & D_2 \\ B_2 & E_2 \\ C_2 & F_2 \end{smallmatrix}\right]} R^3 \xrightarrow{\left[\begin{smallmatrix} z^{r+1} & (z+y)^{r+1} & y^{r+1} \end{smallmatrix}\right]} R \longrightarrow R/I_2 \longrightarrow 0.$$

By [28], Lemma 3.8, $N \approx R(-r-1)/\langle C_1, F_1, C_2, F_2 \rangle$. In what follows we will prove that the Hilbert function of $R/\langle C_1, F_1, C_2, F_2 \rangle$ is nonzero in degree $r-1$, for any positive integer r. In other words

$$\begin{aligned} HF(N, 2r) &= HF(R(-r-1)/\langle C_1, F_1, C_2, F_2 \rangle, 2r) \\ &= HF(R/\langle C_1, F_1, C_2, F_2 \rangle, r-1) \\ &\neq 0. \end{aligned}$$

B. $HF(R/\langle C_1, F_1, C_2, F_2 \rangle, r-1) \neq 0$

In the previous section we saw that the ideals I_1 and I_2 have a special form. First notice that they are symmetric (in terms of the generators) in x and z. So replacing x by z in the forms of C_1 and F_1 we obtain C_2 and F_2. Next observe that we can look at the ideal I_1 as an ideal in $A = \mathbb{R}[x, y]$. Similarly, I_2 is an ideal $A' = \mathbb{R}[y, z]$. Hence $C_1, F_1 \in A$ and $C_2, F_2 \in A'$.

For $i = 1, 2$, the ideal $\langle C_i, F_i\rangle$ is a complete intersection. For example, if $\langle C_1, F_1\rangle$ is not a complete intersection, since $C_1 \neq 0$ and $F_1 \neq 0$, then $codim(\langle C_1, F_1\rangle) = 1$. Therefore, there is a nonunit in A, say d_1, such that $d_1|C_1$ and $d_1|F_1$. Therefore, $d_1|x^{r+1} = B_1F_1 - E_1C_1$ and $d_1|(x+y)^{r+1} = A_1F_1 - D_1C_1$. Hence $codim(\langle x^{r+1}, (x+y)^{r+1}\rangle) = 1$. But this contradicts the fact that $codim(\langle x^{r+1}, (x+y)^{r+1}\rangle) = 2$, as $\{x^{r+1}, (x+y)^{r+1}\}$ is a regular A-sequence. So the ideal $\langle C_1, F_1\rangle$ is a complete intersection. The same argument shows that $\langle C_2, F_2\rangle$ is also a complete intersection.

These observations will simplify our future computations. We need to discuss two cases, depending on if r is odd or even.

1. $r + 1 = 2n$

Let $A = \mathbb{R}[x, y]$ and $I_1 = \langle x^{2n}, (x+y)^{2n}, y^{2n}\rangle$. By [27], Theorem 3.1, a free resolution for I_1 is:

$0 \longrightarrow A(-3n)^2 \xrightarrow{\left[\begin{smallmatrix} A_1 & D_1 \\ B_1 & E_1 \\ C_1 & F_1 \end{smallmatrix}\right]} A(-2n)^3 \xrightarrow{\left[\, x^{2n} \; (x+y)^{2n} \; y^{2n} \,\right]} I_1 \longrightarrow 0$, where $\deg C_1 = \deg F_1 = 3n - 2n = n$. From the observations at the beginning of this section we get the minimal free resolution for $A/\langle C_1, F_1\rangle$:

$$0 \longrightarrow A(-2n) \longrightarrow A(-n)^2 \longrightarrow A \longrightarrow A/\langle C_1, F_1\rangle.$$

Therefore the Hilbert series is $HS(A/\langle C_1, F_1\rangle, t) = \frac{1-2t^n+t^{2n}}{(1-t)^2}$. Hence there exists a monomial x^uy^v of degree $2n-2 = r-1$ which is not in $\langle C_1, F_1\rangle$; an easy computation we can see that this monomial is actually x^{2n-2}.

The previous calculations took place in two variables x and y. Returning to the ring $R = \mathbb{R}[x, y, z]$ and suppose that the above monomial x^uy^v is in $\langle C_1, F_1\rangle +$

$\langle C_2, F_2\rangle = \langle C_1, F_1, C_2, F_2\rangle$. Then there exist $\alpha_1, \beta_1, \alpha_2, \beta_2 \in R$ such that:

$$x^u y^v = \alpha_1 C_1 + \beta_1 F_1 + \alpha_2 C_2 + \beta_2 F_2.$$

In this equation, since for $x = z$ we get $C_1 = C_2$ and $F_1 = F_2$ (see the remarks at the beginning), we obtain an equation in $A = \mathbb{R}[x, y]$:

$$\begin{aligned} x^u y^v &= \alpha_1(x, y, x)C_1 + \beta_1(x, y, x)F_1 + \alpha_2(x, y, x)C_1 + \beta_2(x, y, x)F_1 \\ &= \alpha_1' C_1 + \beta_1' F_1 \end{aligned}$$

So $x^u y^v \in \langle C_1, F_1\rangle$. This contradicts the way we chose $x^u y^v$. Hence there is a monomial of degree $r - 1$ which is not in $\langle C_1, F_1, C_2, F_2\rangle$.

2. $r + 1 = 2n + 1$

For the odd case, the idea is almost identical. Let $A = \mathbb{R}[x, y]$ and $I_1 = \langle x^{2n+1}, (x + y)^{2n+1}, y^{2n+1}\rangle$. Again, by [27], Theorem 3.1, a free resolution for A/I_1 is:

$$0 \longrightarrow \begin{matrix} A(-3n-1) \\ \oplus \\ A(-3n-2) \end{matrix} \xrightarrow{\left[\begin{smallmatrix} A_1 & D_1 \\ B_1 & E_1 \\ C_1 & F_1 \end{smallmatrix}\right]} A(-2n-1)^3 \xrightarrow{[I_1]} A \longrightarrow A/I_1 \longrightarrow 0,$$

where $\deg C_1 = 3n + 1 - (2n + 1) = n$ and $\deg F_1 = 3n + 2 - (2n + 1) = n + 1$. $\langle C_1, F_1\rangle$ is a complete intersection so the minimal free resolution for $A/\langle C_1, F_1\rangle$ is:

$$0 \longrightarrow A(-2n-1) \longrightarrow A(-n) \oplus A(-n-1) \longrightarrow A \longrightarrow A/\langle C_1, F_1\rangle.$$

Therefore the Hilbert series is $HS(A/\langle C_1, F_1\rangle, t) = \frac{1-t^n-t^{n+1}+t^{2n+1}}{(1-t)^2}$. Hence there exists a monomial $x^u y^v$ of degree $2n - 1 = r - 1$ which is not in $\langle C_1, F_1\rangle$. As in the case when r is odd, the same argument gives us that in fact this monomial is not in

$\langle C_1, F_1, C_2, F_2\rangle$.

In conclusion the Hilbert function of $R/\langle C_1, F_1, C_2, F_2\rangle$ is nonzero in degree $r-1$. This is exactly what we wanted to see.

CHAPTER III

$K-$FORMAL ARRANGEMENTS

A. Introduction

In what follows we adopt all the notation from [4]. Let $\mathcal{A}$ be an arrangement of n hyperplanes in a vector space V over a field $\mathbb{K}$. For each $H \in \mathcal{A}$ we fix the defining form $\alpha_H \in V^*$.

Define a map $\phi : E(\mathcal{A}) := \oplus_{H\in\mathcal{A}}\mathbb{K}e_H \to V^*$, by $\phi(e_H) = \alpha_H$, where $E(\mathcal{A})$ is the vector space with basis $\{e_H\}$.

Let $F(\mathcal{A})$ be the kernel of this map. Then $\dim F(\mathcal{A}) = n - r(\mathcal{A})$ where $r(\mathcal{A})$ is the rank of $\mathcal{A}$. The vector space $F(\mathcal{A})$ describes which linear forms are linearly dependent, as well as the dependency coefficients (up to scalar multiplication). We will refer to elements of $F(\mathcal{A})$ as *relations*.

Let $F_2(\mathcal{A})$ be the subspace of $F(\mathcal{A})$ generated by the relations corresponding to dependencies of exactly 3 linear forms.

Definition III.1. $\mathcal{A}$ is formal iff $F(\mathcal{A}) = F_2(\mathcal{A})$.

Definition III.2. For $3 \leq k \leq r(\mathcal{A})$, recursively define $R_k(\mathcal{A})$ to be the kernel of the map

$$\pi_{k-1} = \pi_{k-1}(\mathcal{A}) : \bigoplus_{X\in L, r(X)=k-1} R_{k-1}(\mathcal{A}_X) \to R_{k-1}(\mathcal{A}),$$

where L is the lattice of intersections of $\mathcal{A}$ and π_{k-1} is the sum of the inclusion maps $R_{k-1}(\mathcal{A}_X) \hookrightarrow R_{k-1}(\mathcal{A})$. We identify $R_2(\mathcal{A})$ with $F(\mathcal{A})$.

To simplify notation, for $k \geq 2$ we will denote with $D_k = D_k(\mathcal{A})$ the vector space $\bigoplus_{X\in L, r(X)=k} R_k(\mathcal{A}_X)$.

Definition III.3. We define

1. An arrangement is 2-formal if it is formal.
2. For $k \geq 3$, $\mathcal{A}$ is $k-$formal iff it is $(k-1)-$formal and the map $\pi_k : D_k \to R_k(\mathcal{A})$ is surjective.

Lemma III.1. *For any arrangement $\mathcal{A}$, the following sequence of vector spaces and maps form a complex:*

$$D_\bullet : 0 \longrightarrow \cdots \xrightarrow{d_3} D_2 \xrightarrow{d_2} D_1 \xrightarrow{d_1} D_0 \longrightarrow 0,$$

where $D_0 = V^*$, $D_1 = E(\mathcal{A})$ and for $k \geq 2$, D_k are the spaces from the notations above. Also, $d_1 = \phi$ and $d_k : D_k \to D_{k-1}, d_k = \pi_k$ for $k \geq 2$.

Proof. We have $d_k(D_k) = \pi_k(D_k) \subseteq R_k(\mathcal{A}) = ker(\pi_{k-1}) \subseteq D_{k-1}$. So d_k is well defined. Also, $d_{k-1} \circ d_k(v) = \pi_{k-1}(\pi_k(v)) = 0$ for any $v \in D_k$, as $\pi_k(v) \in R_k(\mathcal{A}) = ker(\pi_{k-1})$. So, indeed we have a complex. □

Proposition III.1. *$\mathcal{A}$ is $k-$formal iff $H_i(D_\bullet) = 0$ for every $i = 1, \ldots, k-1$*

Proof. π_l is surjective iff $\forall w \in R_l(\mathcal{A})$ there exists $v \in D_l$ such that $\pi_l(v) = w$.

We have $R_l(\mathcal{A}) = ker(\pi_{l-1}) = ker(d_{l-1})$ and $w = \pi_l(v) = d_l(v) \in Im(d_l)$. So we get $ker(d_{l-1}) \subseteq Im(d_l)$ which give us $H_{l-1}(D) = 0$. □

Example III.1. In this example we will discuss [4], Example 5.1., in terms of the homology of the above complex. We must specify that all the computations are already done in [4], and we are just translating into topological language.

$\mathcal{A}$ is a real essential arrangement of rank 4 consisting of 10 hyperplanes, defined by the vanishing of the following linear forms:

$$
\begin{aligned}
\alpha_1 &= x_3 \\
\alpha_2 &= x_3 - x_4 \\
\alpha_3 &= x_2 \\
\alpha_4 &= x_2 + x_3 - 2x_4 \\
\alpha_5 &= x_1 \\
\alpha_6 &= x_1 + x_3 - 2x_4 \\
\alpha_7 &= x_2 + 2x_3 - 2x_4 \\
\alpha_8 &= x_1 + 2x_3 - 2x_4 \\
\alpha_9 &= x_1 + x_2 + x_3 - 2x_4 \\
\alpha_{10} &= x_4
\end{aligned}
$$

So $D_0 = \mathbb{R}^4$, $D_1 = \mathbb{R}^{10}$ and the map $d_1 : D_1 \longrightarrow D_0$ is just the map ϕ and has rank 4. Therefore $ker(d_1)$ has dimension $10 - 4 = 6$.

We have 7 nondegenerate rank 2 elements in $L(\mathcal{A})$ and each is an intersection of exactly 3 hyperplanes. So we have 7 relations of length 3:

$$
\begin{aligned}
\alpha_1 - \alpha_2 - \alpha_{10} &= 0 \\
\alpha_1 + \alpha_4 - \alpha_7 &= 0 \\
\alpha_1 + \alpha_6 - \alpha_8 &= 0 \\
2\alpha_2 + \alpha_3 - \alpha_7 &= 0 \\
2\alpha_2 + \alpha_5 - \alpha_8 &= 0 \\
\alpha_3 + \alpha_6 - \alpha_9 &= 0 \\
\alpha_4 + \alpha_5 - \alpha_9 &= 0
\end{aligned}
$$

Therefore $D_2 = \mathbb{R}^7$. The matrix of the map $d_2 : D_2 \longrightarrow D_1$ is exactly the matrix in [4], page 61

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

and it has rank 6. So $\dim Im(d_2) = 6$ and $\dim ker(d_2) = 7 - 6 = 1$.

Also in [4] all the elements of rank 3 from $L(\mathcal{A})$ are listed: $\{1,2,9,10\}$, $\{3,6,9,10\}$, $\{4,5,9,10\}$, $\{1,3,6,8,9\}$, $\{1,4,5,7,9\}$, $\{1,4,6,7,8\}$, $\{2,3,5,7,8\}$, $\{2,3,6,7,9\}$, $\{2,4,5,8,9\}$, $\{3,4,5,6,9\}$, $\{1,2,3,4,7,10\}$, $\{1,2,5,6,8,10\}$.

If X is such an element (with $r(X) = 3$), then $R_3(\mathcal{A}_X) \neq 0$ means that there is at least a relation among the relations of length 3 of elements of rank 2 in $L(\mathcal{A}_X)$. The nondegenerate rank 2 elements in $L(\mathcal{A}_X)$ are nondegenerate rank 2 elements in $L(\mathcal{A})$ and these are listed above. It is not difficult to check which are the relations of length 3 for each rank 3 element in $\mathcal{A}$. For reference, these are listed in the chart on page 62 in [4]. Also, there is no problem to check that for each $r(X) = 3$, the length 3 relations are linearly independent. Therefore we conclude that $D_3 = 0$.

So the complex we get is:

$$D_\bullet : 0 \longrightarrow \mathbb{R}^7 \longrightarrow \mathbb{R}^{10} \longrightarrow \mathbb{R}^4 \longrightarrow 0$$

with homology: $H_1(D_\bullet) = 0$ and $H_2(D_\bullet) = 1$. So $\mathcal{A}$ is formal, but not 3-formal.

B. Graphic Arrangements

Let G be a connected graph on vertices $[n] = \{1, \ldots, n\}$ with no loops or multiple edges. We will denote with $H_\bullet(\Delta)$ the homology of the chain complex of the flag complex associated to G (see Example I.4).

By definition (see Example I.5), the graphic arrangement associated to G is $\mathcal{A} = \mathcal{A}_G = \{ker\{\alpha_{ij}\} | \alpha_{ij} = x_i - x_j, i < j \text{ and } [ij] \text{ is an edge in } G\}$. Note that $\mathcal{A}$ is an arrangement in $V = \mathbb{K}^{a_0}$ of rank $a_0 - 1$ (if G is connected) and consists of a_1 (= the number of edges in G) hyperplanes.

Notice that from the beginning we fixed the defining forms α_{ij}. To be consistent with notation, $e_{ij}, i < j$ will be the symbols in $E(\mathcal{A})$ (i.e., $\phi(e_{ij}) = \alpha_{ij}$). With these, we can identify $D_1 = E(\mathcal{A})$ with C_1 by $e_{ij} \leftrightarrow [ij]$ for $i < j$.

If we fix the form of the elements in the basis of D_i's and with proper notations of those, the correspondence between the two complexes will become natural. The next lemma will do this, but before we state and prove it here is the flavor of it:

For $X \in L$, let G_X be the subgraph of G built on the edges corresponding to the hyperplanes in X.

We have $D_2 = \oplus_{X \in L_2} R_2(\mathcal{A}_X)$. Suppose for an $X \in L_2$ we have $R_2(\mathcal{A}_X) = F(\mathcal{A}_X) \neq 0$. This means that we must have a dependency (relation) among some of the linear forms corresponding to some edges in G_X. But this translates in the fact that G_X contains a cycle. If the length of this cycle is ≥ 4, then the linear forms corresponding to 3 consecutive edges in the cycle are linearly independent. This contradicts the fact that $rk(X) = 2$. So G_X contains a triangle. If we have an extra edge in G_X, beside those from the triangle, then the linear form of this extra edge and the linear forms associated to two of the edges of the triangle are linearly independent. Again we get a contradiction with the fact that $rk(X) = 2$. So $G_X =$ a triangle. So each nonzero summand of D_2 corresponds to a triangle in G. The converse of this statement is obvious.

Lemma III.2. *(The Recursive Identification Lemma) Let $X \in L$ with $r(X) = l, l \geq 2$. Then $R_l(\mathcal{A}_X) \neq 0$ iff G_X is a K_{l+1} subgraph of G. More, $\dim R_l(\mathcal{A}_X) = 1$ and if $G_X = [i_1 i_2 \cdots i_{l+1}], i_1 < i_2 < \cdots i_{l+1}$, then we can pick a 'special' basis element of $R_l(\mathcal{A}_X)$ to be the relation on the special elements corresponding to the K_l subgraphs of G_X: $r_{i_2 \cdots i_{l+1}} - r_{i_1 i_3 \cdots i_{l+1}} + \cdots + (-1)^l r_{i_1 i_2 \cdots i_l}$. This element is denoted with $r_{i_1 i_2 \cdots i_{l+1}}$.*

Proof. Suppose $R_l(\mathcal{A}_X) \neq 0$. We will use induction on l.

For $l = 2$ we already seen this case above.

Suppose $l \geq 3$.

By definition, we have $R_l(\mathcal{A}_X) = ker(\pi_{l-1})$, where

$$\pi_{l-1} : D_{l-1}(\mathcal{A}_X) = \bigoplus_{Y \in L(\mathcal{A}_X), r(Y) = l-1} R_{l-1}((\mathcal{A}_X)_Y) \longrightarrow R_{l-1}(\mathcal{A}_X).$$

The induction hypothesis is telling that for each $Y \in L(\mathcal{A}_X), r(Y) = l - 1$ such that $R_{l-1}((\mathcal{A}_X)_Y) \neq 0, G_Y = [i_1 i_2 \cdots i_l]$ is a K_l subgraph of G_X and $\dim R_{l-1}((\mathcal{A}_X)_Y) = 1$ with $r_{i_1 i_2 \cdots i_l}$ 'special' basis element of $R_{l-1}((\mathcal{A}_X)_Y)$.

From this we get first that $\dim D_{l-1}(\mathcal{A}_X) =$ the number of K_l subgraphs of G_X.

The condition $R_l(\mathcal{A}_X) \neq 0$ implies that since $R_l(\mathcal{A}_X) \subseteq D_{l-1}(\mathcal{A}_X)$, G_X has at least one K_l subgraph.

If G_X has just one K_l subgraph, then $R_l(\mathcal{A}_X) = D_{l-1}(\mathcal{A}_X)$. But π_{l-1} is a sum of inclusions, and in this particular case it will be exactly an inclusion. So we get that $R_l(\mathcal{A}_X) = ker(\pi_{l-1}) = 0$, which is a contradiction. Therefore, G_X has at least two K_l subgraphs.

Take two of them K_l^1 and K_l^2, and first suppose they do not share any vertex. Let

$v \in K_l^1$ and $w \in K_l^2$ be two vertices of G_X. Through v pass exactly $l-1$ edges and the corresponding linear forms $\alpha_1, \ldots, \alpha_{l-1}$ are linearly independent. Let's take two edges $[w, w_1]$ and $[w, w_2]$ of K_l^2 and let β_1 and β_2 be the corresponding linear forms. Then, $\alpha_1, \ldots, \alpha_{l-1}, \beta_1, \beta_2$ are linearly dependent if at least one of the vertices $\{w, w_1, w_2\}$ is a vertex in K_l^1. Contradiction. Therefore, $\alpha_1, \ldots, \alpha_{l-1}, \beta_1, \beta_2$ are linearly independent. But this will contradict $r(\mathcal{A}_X) = l$.

Hence, K_l^1 and K_l^2 have at least a common vertex v. Suppose w_1, w_2 are two vertices of K_l^2 but not of K_l^1. Then, through v pass at least $l+1$ edges: $l-1$ from K_l^1 and $[v, w_1]$, $[v, w_2]$ from K_l^2. The corresponding linear forms are linearly independent and again we obtain a contradiction with the fact that $r(\mathcal{A}_X) = l$.

The conclusion of all of above is that any two distinct K_l subgraphs of G_X have exactly $l-1$ vertices in common. $(*)$

Suppose G_X has exactly two K_l subgraphs: $[1, 2, \ldots, l-1, l]$ and $[1, 2, \ldots, l-1, l+1]$. Let $r \in R_l(\mathcal{A}_X), r \neq 0$. Then $r = r_{1,2,\ldots,l-1,l} + b r_{1,2,\ldots,l-1,l+1}$ for some $b \in \mathbb{K} - \{0\}$. We have $\pi_{l-1}(r) = 0$ in $D_{l-2}(\mathcal{A}_X)$. So we get a relation on the 'special' basis elements of $D_{l-2}(\mathcal{A}_X)$:

$$0 = (r_{2,\ldots,l-1,l} - r_{1,3,\ldots,l-1,l} + \cdots + (-1)^{l-1} r_{1,2,\ldots,l-1})$$

$$+b(r_{2,\ldots,l-1,l+1} - r_{1,3,\ldots,l-1,l+1} + \cdots + (-1)^{l-1} r_{1,2,\ldots,l-1}).$$

Observe that this equation is impossible.

So G_X has at least three distinct K_l subgraphs: $K_l^1 = [1, 2, \ldots, l-1, l]$, $K_l^2 = [1, 2, \ldots, l-1, l+1]$ and K_l^3. If both l and $l+1$ are vertices in K_l^3, then l and $l+1$ are connected in G_X, so G_X contains a K_{l+1} subgraph. If, for example, $l \notin K_l^3$, then from $(*)$ and since $K_l^i, i = 1, 2, 3$ are distinct we get that $K_l^3 = [1, 2, \ldots, l-1, l+2]$, for some other vertex $l+2$ in G_X. Observe that through the vertex 1 pass at least

$l+1$ edges of G_X: $[1,2],[1,3],\ldots,[1,l-1],[1,l],[1,l+1],[1,l+2]$. The corresponding linear forms of these edges are linearly independent so we get a contradiction with the fact that $r(\mathcal{A}_X)=l$.

We can conclude that G_X contains a K_{l+1} subgraph. Now, if there exists an extra edge of G_X not on this K_{l+1}, then the corresponding linear form of this edge together with the corresponding linear forms of the edges passing through any vertex of the K_{l+1} subgraph will form a linearly independent set of $l+1$ elements. Again we get a contradiction with the fact that $r(\mathcal{A}_X)=l$. So G_X is a K_{l+1}.

With this, G_X has exactly $l+1$ K_l subgraphs. These subgraphs will give us the 'special' elements of $D_{l-1}(\mathcal{A}_X)$: $r_{2,\ldots,l+1}, r_{1,3,\ldots,l+1},\ldots,r_{1,2,\ldots,l}$. The only relation on these elements is exactly the 'special' element in $R_l(\mathcal{A}_X)$:

$$r_{2,\ldots,l+1}-r_{1,3,\ldots,l+1}+\cdots+(-1)^l r_{1,2,\ldots,l}.$$

We denote this element with $r_{1,2,\ldots,l,l+1}$ and he is forming the basis for $R_l(\mathcal{A}_X)$.

For the converse, it is obvious that if G_X is a K_{l+1}, then $R_l(\mathcal{A}_X)\neq 0$ and even more, $\dim R_l(\mathcal{A}_X)=1$. $\square$

With this lemma we can identify easily the two complexes. The way we pick the special basis elements will give us the same matrices for the differentials of the two complexes and, hence, with Proposition III.1, we have proved the following:

Proposition III.2. *Let G be a connected graph. $\mathcal{A}_G$ is $k-$formal if and only if $H_i(\Delta)=0$ for every $i=1,\ldots,k-1$.*

Note that from this proposition we get that in the graphic arrangement case, $k-$formality depends only on combinatorics, contrary to the case of line arrangements (see Yuzvinsky's example, [32]).

Example III.2. We conclude with an example of a formal graphic arrangement which is not 3-formal. Consider the graph G in the Figure 3 below:

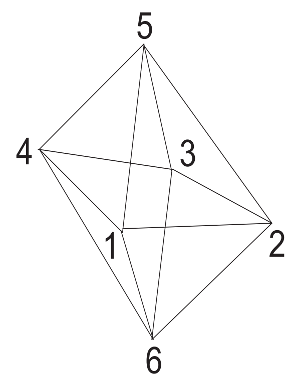

Fig. 3. 2-formal but not 3-formal graphic arrangement.

The associated flag complex Δ is the boundary complex of an octahedron on the same vertices and edges. The associated chain complex of Δ is:

$$0 \longrightarrow \mathbb{K}^8 \xrightarrow{f_2} \mathbb{K}^{12} \xrightarrow{f_1} \mathbb{K}^6 \longrightarrow 0,$$

where, if we order the basis lexicographically we have:

$$f_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

and

$$f_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since G is connected, $\dim H_0(\Delta) = 1$. So $rk(f_1) = 6 - 1 = 5$. Therefore, $\dim ker(f_1) = 12 - 5 = 7$.

Every 4-cycle in G is a linear combination of 3-cycles. So $\mathcal{A}_G$ is formal (2-formal). By the proposition above, $\dim H_1(\Delta) = 0$ and with this we get $rk(f_2) = 7$. Therefore, $\dim ker(f_2) = 8 - 7 = 1$. So we get $\dim H_2(\Delta) = 1$. Hence $\mathcal{A}_G$ is not 3-formal.

CHAPTER IV

CONFIGURATIONS OF SMOOTH RATIONAL CURVES IN $\mathbb{P}^2$

A. Introduction

In Chapter 1, the module of derivations $D(\mathcal{A})$ was defined for a hyperplane arrangement $\mathcal{A}$. $\mathcal{A}$ is free exactly when $D(\mathcal{A})$ is a free module. In this chapter, we restrict to $\mathbb{P}^2$, but broaden the class of curves which make up the arrangement. In particular, suppose $\mathcal{C} = \bigcup_{i=1}^n C_i$ where each C_i is a smooth rational plane curve, such that $\mathcal{C}$ has only ordinary singularities; call such a collection a *smooth rational curve (SRC) arrangement*. Throughout this chapter, $S = \mathbb{C}[x, y, z]$.

Define the module of logarithmic derivation

$$D(\mathcal{C}) = \{\theta \in Der_{\mathbb{K}}(S) | \theta(C_i) \in \langle C_i \rangle, \forall i = 1, \ldots, n\},$$

where by C_i is the equation of the smooth rational plane curve C_i. We say that $\mathcal{C}$ is free iff $D(\mathcal{C})$ is free. Proposition I.2 and Theorem I.1 have natural extensions to SRC arrangements.

Example IV.1. For the SRC arrangement in Figure 4, $D(\mathcal{C}) \simeq S(-1) \oplus S(-2) \oplus S(-5)$.

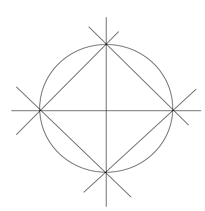

Fig. 4. Free SRC arrangement.

For a hyperplane arrangement $\mathcal{A}$, the intersection lattice $L_{\mathcal{A}}$ of $\mathcal{A}$ consists of the intersections of the elements of $\mathcal{A}$; the rank of $x \in L_{\mathcal{A}}$ is simply the codimension of x. V is the lattice element $\hat{0}$; the rank one elements are the hyperplanes themselves. $\mathcal{A}$ is called *essential* if rank $L_{\mathcal{A}} = \dim V$. The reason we introduce the intersection lattice is that we can obtain the Poincare polynomial of the complement just from the information in this lattice.

Definition IV.1. The Möbius function $\mu : L_{\mathcal{A}} \longrightarrow \mathbb{Z}$ is defined by

$$\begin{aligned} \mu(\hat{0}) &= \quad 1 \\ \mu(t) &= -\sum_{s<t} \mu(s), \text{ if } \hat{0} < t \end{aligned}$$

We now restrict to the case that V is complex. A foundational result is that the Poincaré polynomial of $X = V \setminus \mathcal{A}$ is purely combinatorial; in particular

$$P(X,t) = \sum_{x \in L_{\mathcal{A}}} \mu(x) \cdot (-t)^{rank(x)}.$$

If $\mathcal{A} \subseteq \mathbb{C}^3$ is central, then $\mathcal{A}$ also defines a set of lines in $\mathbb{P}^2$, and obviously $X = \mathbb{C}^3 \setminus \mathcal{A} \simeq \mathbb{C}^* \times \widetilde{X}$, where $\widetilde{X}$ is the complement of the corresponding arrangement of lines in $\mathbb{P}^2$. Hence

$$P(\widetilde{X},t) = 1 + (n-1)t + (\sum_{\substack{x \in L_{\mathcal{A}} \\ rank(x)=2}} \mu(x) - n + 1)t^2.$$

It follows from Terao's theorem (see I.2) that if $D_0(\mathcal{A}) \simeq S(-a) \oplus S(-b)$, then $P(\widetilde{X},t) = (1+at)(1+bt)$. This can be generalized to line arrangements which are not free, using the Chern polynomial ([22]).

The motivating question of this chapter is: *what happens if the arrangement of lines is replaced with an SRC arrangement?*

1. Rational curve arrangements

In [5], Cogolludo-Agustín studies the complement of an arrangement of rational curves in $\mathbb{P}^2$, where the individual curves can have singularities, and can meet non-transversally. The main result is that the cohomology ring of the complement to a rational curve arrangement is generated by logarithmic 1 and 2-forms and its structure depends on a finite number of invariants of the curve. One fact is that if $\widetilde{X}$ is the complement of an arrangement of n irreducible curves in $\mathbb{P}^2$, then

$$h^1(\widetilde{X}, \mathbb{C}) = n - 1$$

$$h^2(\widetilde{X}, \mathbb{C}) = 1 + \sum_{p \in sing(\mathcal{C})} (r_p - 1) - \sum_1^n (\chi(\hat{C}_i) - 1),$$

where r_p is the number of branches passing thru p, and $\hat{C}_i$ is the normalization of C_i. Since we are assuming only ordinary singularities and that all the C_i are smooth and rational, we have that

$$h^2(\widetilde{X}, \mathbb{C}) = \sum_{p \in L_2(\mathcal{C})} (r_p - 1) - |\mathcal{C}| + 1,$$

where the intersection poset $L(\mathcal{C})$ is defined precisely as for a linear arrangement (typically, $L(\mathcal{C})$ is only a poset, not a lattice).

2. Milnor and Tjurina number

A crucial distinction between line and curve arrangements, even in our simple setting, is the difference between the Milnor and Tjurina numbers at a singularity. Let $\mathcal{C}$ be a reduced (but not irreducible) curve in $\mathbb{P}^2$, $V(F) = \mathcal{C} = \bigcup C_i$. Suppose $\mathcal{C}$ has a singularity at $(0:0:1)$, $f = F(x, y, 1)$, and let $\mathbb{C}\{x, y\}$ be the ring of convergent power series.

Definition IV.2. The Milnor number μ of $V(f)$ is $\dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$.

Let r be the number of irreducible components of $\mathcal{C}$ at $(0,0)$. An *infinitely near point* of $(0,0)$ is a point in $\widetilde{\mathcal{C}}$ which maps to $(0,0)$. Then the geometric content of μ is that:

$$\mu = \sum_i \nu_i(\nu_i - 1) - r + 1,$$

where ν_i runs thru the multiplicities of the strict preimages of all infinitely near points of $(0,0)$. In the setting of line configurations and curves with only ordinary singularities, a single blowup suffices to separate the points, so we only need ν_0, the multiplicity of the curve at $(0,0)$. If there are n distinct tangents at $(0,0)$, then $\mu = n(n-1) - n + 1 = (n-1)^2$, hence when $\mathcal{C}$ is a line arrangement, the degree of the Jacobian ideal of the defining polynomial is the sum of the Milnor numbers at the singularities. This is not true for general curve arrangements. From Fulton ([**?**], p. 42),

$$\deg Z = \sum_{p \in V(J)} \text{length } \mathcal{O}_p / J\mathcal{O}_p.$$

After a translation, we may assume that F has no singularities along $z = 0$. So we also have that the degree of Z is $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f \rangle$. From [6], Corollary 4.2.5, this dimension is equal to $\sum_{p \in V(J)} \text{length } \mathcal{O}_p / J\mathcal{O}_p$.

Definition IV.3. The Tjurina number τ of $V(f)$ is $\dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f \rangle$.

Corollary IV.1. *So for a general curve arrangement, the degree of the Jacobian ideal is the sum of the Tjurina numbers at the singularities.*

Example IV.2. We give an example where the Tjurina and Milnor numbers differ:

Let $\mathcal{C}$ be the union of one conic and four lines (see Figure 5 below):

$$\begin{aligned} C &= x^2 - xz + y^2 - yz = 0 \\ L_1 &= \qquad x = 0 \\ L_2 &= \qquad y = 0 \\ L_3 &= \quad x - y = 0 \\ L_4 &= \quad x - 2y = 0 \end{aligned}$$

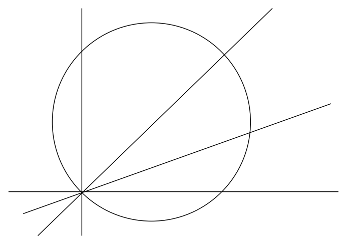

Fig. 5. SRC arrangement with $\mu = 16$ and $\tau = 15$ at $(0:0:1)$.

$\mathcal{C}$ has 5 singular points, all ordinary. At four of the singular points, only two branches of $\mathcal{C}$ meet, while at the fifth point all 5 branches of $\mathcal{C}$ meet. The degree of the Jacobian ideal is 19 and the sum of Milnor numbers is 20; at $(0:0:1)$ we have $\mu = 16$ but $\tau = 15$.

In [21], Saito proved that if f is a convergent power series with isolated singularity at the origin, then f is in the ideal generated by the partial derivatives if and only if f is quasihomogeneous after a holomorphic change of coordinates (in the two variable case, this was first proved by Reiffen ([19])). In particular, $\mu = \tau$ at a singular point iff we can change coordinates to make f quasihomogeneous.

3. Criteria for freeness

There are two fundamental tools that can be used to prove that a line arrangement is free. The first method is based on an inductive operation known as deletion-restriction: given an arrangement $\mathcal{A}$ and choice of hyperplane $H \in \mathcal{A}$, set

$$\mathcal{A}' = \mathcal{A} \setminus H \text{ and } \mathcal{A}'' = \mathcal{A}|_H.$$

The collection $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ is called a *triple*, and a triple yields (see Proposition 4.45 of [17]) a left exact sequence

$$0 \longrightarrow D(\mathcal{A}')(-1) \xrightarrow{\cdot H} D(\mathcal{A}) \longrightarrow D(\mathcal{A}'').$$

For a triple with $\mathcal{A} \subseteq \mathbb{P}^2$, more is true (see [24]): after pruning the Euler derivations and sheafifying, there is an exact sequence

$$0 \longrightarrow \mathcal{D}_0{}'(-1) \longrightarrow \mathcal{D}_0 \longrightarrow i_*\mathcal{D}_0{}'' \longrightarrow 0, \tag{4.1}$$

where $i : H \hookrightarrow \mathbb{P}^2$; $i_*\mathcal{D}_0{}'' \simeq \mathcal{O}_H(1 - |\mathcal{A}''|)$.

In [29], Terao showed that freeness of a triple is related via:

Theorem IV.1. *(Addition-Deletion) Let $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ be a triple. Then any two of the following imply the third*

- $D(\mathcal{A}) \simeq \oplus_{i=1}^{n} S(-b_i)$
- $D(\mathcal{A}') \simeq S(-b_n + 1) \oplus_{i=1}^{n-1} S(-b_i)$
- $D(\mathcal{A}'') \simeq \oplus_{i=1}^{n-1} S(-b_i)$

Theorem IV.1 applies in general, not just to arrangements in $\mathbb{P}^2$. A smooth conic is intrinsically a $\mathbb{P}^1$, so it is natural to ask if SRC arrangements which admit a short exact sequence similar to (4.1) have an addition-deletion theorem; we tackle this in the next two sections.

A second criterion for freeness is special to the case of line arrangements; to state it we need to define freeness for *multiarrangements*. A multiarrangement $(\mathcal{A}, \mathbf{m})$ is an arrangement together with a multiplicity m_i for each hyperplane. The module of derivations consists of θ such that $l_i^{m_i} | \theta(l_i)$. As shown by Ziegler in [35], freeness of multiarrangements is not combinatorial; for recent progress see [31].

Theorem IV.2. *(Yoshinaga's multiarrangement criterion [33])* $\mathcal{A} \subseteq \mathbb{P}^2$ *is free iff* $\pi(\mathcal{A}, t) = (1+t)(1+at)(1+bt)$ *and* $\forall H \in \mathcal{A}$ *the multiarrangement* $\mathcal{A}|_H$ *has minimal generators in degree* a *and* b*.*

The main result of this chapter is an addition-deletion theorem for SRC arrangements with quasihomogeneous singularities; the freeness of Example IV.1 is explained by the theorem. As one application, we show that a free SRC arrangement, when restricted to different lines, can yield multiarrangements with different exponents; hence any version of Theorem IV.2 for SRC arrangements will be quite subtle.

B. Addition-deletion for a line

Let $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ be a triple of SRC arrangements in $\mathbb{P}^2$, where $\mathcal{C}' = \mathcal{C} \setminus \{L\}$, $\mathcal{C}'' = \mathcal{C}|_L$ and $L \in \mathcal{C}$ *is a line*. We begin by examining some examples:

Example IV.3. Let $\mathcal{C}'$ (see Figure 6 below) be the union of:

$$\begin{aligned} C_1 &= x^2 - xz + 5y^2 - 5yz = 0 \\ C_2 &= x^2 + 2y^2 - xz - 2yz = 0 \\ L_1 &= \qquad\qquad x = 0 \\ L_2 &= \qquad\qquad y = 0 \\ L_3 &= \qquad x + y - z = 0 \end{aligned}$$

$D(\mathcal{C}')$ is free with exponents $\{1, 2, 4\}$, and the degree of the Jacobian ideal is 28, which is equal to the sum of the Milnor numbers at the intersection points.

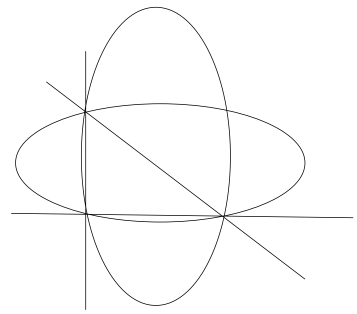

Fig. 6. Free, quasihomogeneous SRC arrangement with exponents $\{1, 2, 4\}$.

Therefore at each singular point $\tau = \mu$. If we restrict to any line, the corresponding multiarrangement has 2 points of multiplicity 3, and it follows from [31] that the exponents are $\{3, 3\}$. Hence the obvious generalization of Yoshinaga's criterion does not hold.

Example IV.4. Let $\mathcal{C}_1 = \mathcal{C}' \cup L_4 = \{x - y = 0\}$ (see Figure 7). The degree of the

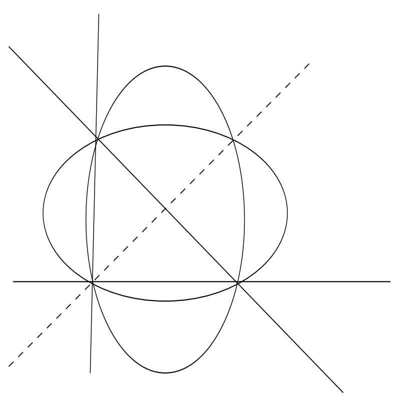

Fig. 7. Free, quasihomogeneous SRC arrangement with exponents $\{1, 2, 5\}$.

Jacobian ideal is 39, which is equal to the sum of Milnor numbers at the points. It can be shown that $D(\mathcal{C}_1)$ is free with exponents $\{1, 2, 5\}$.

Example IV.5. If instead we take $\mathcal{C}_2 = \mathcal{C}' \cup L_4 = \{x - 2y = 0\}$ (see Figure 8), then $\mathcal{C}_2$ is free with exponents $\{1, 3, 4\}$. The degree of the Jacobian ideal is 37, whereas the sum of the Milnor numbers is 38; the singularity at $(0 : 0 : 1)$ has $\tau = 15$ and $\mu = 16$.

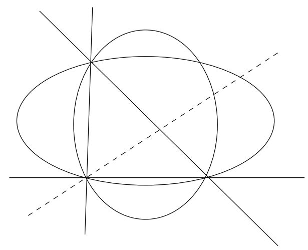

Fig. 8. Free, not quasihomogeneous SRC arrangement with exponents $\{1, 3, 4\}$.

For SRC arrangements similar to $\mathcal{C}_1$, there is a version of the addition-deletion theorem:

Definition IV.4. A triple $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ of SRC arrangements is called *quasihomogeneous* if $\tau = \mu$ at each singular point of $\mathcal{C}'$ and $\mathcal{C}$.

Theorem IV.3. *Let $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ be a quasihomogeneous triple with $|L \cap \mathcal{C}| = |\mathcal{C}''| = k$. The following are equivalent:*

1. *$\mathcal{C}'$ is free with exponents $\{1, k-1, a\}$.*
2. *$\mathcal{C}$ is free with exponents $\{1, k-1, a+1\}$.*

Examples IV.3 and IV.4 illustrate the theorem. Before giving the proof we point out a nice application.

Corollary IV.2. *A free SRC arrangement, when restricted to a line, can yield different multiarrangements.*

Proof. In Example IV.4, add a line L through the point $(1:1:1)$, say $L = x - \alpha y + (\alpha - 1)z = 0$, with $\alpha \notin \{0, 1, -5, -2, \infty\}$. These choices mean that L is not tangent to the conics, and misses all singularities save $(1:1:1)$. The new arrangement is quasihomogeneous, and L meets $\mathcal{C}_1$ in 6 points. By Theorem 2.5, the new arrangement is free with exponents $\{1, 3, 5\}$.

Restrict this new arrangement to the line $L_3 = \{x + y - z = 0\}$. After a change of coordinates, we obtain a multiarrangement with defining polynomial

$$x^3 y^3 (x - y)(\alpha x - y).$$

This is exactly Ziegler's example from [35]: $\alpha = -1$ gives exponents $\{3, 5\}$, and for $\alpha \neq -1$, the exponents are $\{4, 4\}$. $\square$

Before giving the proof of Theorem IV.3, we need some preliminaries.

Lemma IV.1. *Let $L = \{x = 0\}$. Then the maps $p : D(\mathcal{C}') \longrightarrow D(\mathcal{C})$, $p(\theta) = x\theta$ and $q : D(\mathcal{C}) \longrightarrow D(\mathcal{C}'')$, $q(a\partial_x + b\partial_y + c\partial_z) = b(0, y, z)\partial_y + c(0, y, z)\partial_z$ are well defined and yield an exact sequence:*

$$0 \longrightarrow D(\mathcal{C}') \longrightarrow D(\mathcal{C}) \longrightarrow D(\mathcal{C}'').$$

Proof. Let $f = xf'$ be the defining polynomial of $\mathcal{C}$, where f' is the defining polynomial of $\mathcal{C}'$. Then the defining polynomial of $\mathcal{C}''$ is $f'' = \sqrt{f'|_{x=0}}$. If $\theta' \in D(\mathcal{C}')$, then $\theta'(f') = Pf'$ for some $P \in S$;

$$p(\theta')(f) = x\theta'(xf') = x(f'\theta'(x) + x\theta'(f')) \in \langle f \rangle.$$

So p is well defined and injective. Let $\theta = a\partial_x + b\partial_y + c\partial_z \in D(\mathcal{C})$. Then $\theta(x) = a \in \langle x \rangle$, so $a = xa'$. If $\theta \in ker(q)$, then $b = xb'$ and $c = xc'$, hence $\theta = x\theta'$, where $\theta' = a'\partial_x + b'\partial_y + c'\partial_z$. Because $\theta \in D(\mathcal{C})$, $\theta(f') = x\theta'(f') \in \langle f' \rangle$. Since x and f' are relatively prime, we get that $\theta'(f') \in \langle f' \rangle$, which implies that $\theta \in Im(p)$.

It remains to show is that q is well defined. For suitable $u_i, v_i \in \mathbb{C}$ and $m_i \in \mathbb{Z}$ we have that

$$f'|_{x=0} = \prod_i (u_i y + v_i z)^{m_i}, \text{ so } f'' = \prod_i (u_i y + v_i z).$$

Let L' be a line in $\mathcal{C}'$ defined by the vanishing of $t_i x + u_i y + v_i z = 0$ for some i and $t_i \in \mathbb{C}$, and let $\theta = a\partial_x + b\partial_y + c\partial_z \in D(\mathcal{C})$. Then $\theta(L') \in \langle L' \rangle$, so evaluating at $x = 0$ and using the earlier observation that $a = xa'$, we find $(b(0, y, z)\partial_y + c(0, y, z)\partial_z)(u_i y + v_i z) \in \langle u_i y + v_i z \rangle$.

Now suppose C is a conic in $\mathcal{C}'$; after a change of coordinates we may assume C intersects $L = x = 0$ in the points $(0 : 0 : 1)$ and $(0 : u : v)$. Then $C = xA + y(vy - uz)$ and $C|_{x=0} = y(vy - uz)$, where A is some linear form. We have $\theta(C) =$

$$a(A + x\partial_x(A)) + x(b\partial_y(A) + c\partial_z(A)) + b\partial_y(y(vy - uz)) + c\partial_z(y(vy - uz)) \in \langle C \rangle.$$

Evaluating at $x = 0$ and again using that $a = xa'$ we find

$$(b(0, y, z)\partial_y + c(0, y, z)\partial_z)(y(vy - uz)) \in \langle y(vy - uz) \rangle.$$

Since y and $vy - uz$ are relatively prime we get that

$$(b(0, y, z)\partial_y + c(0, y, z)\partial_z)(y) \in \langle y \rangle$$
$$(b(0, y, z)\partial_y + c(0, y, z)\partial_z)(vy - uz) \in \langle vy - uz \rangle.$$

This shows that for each factor $u_i y + v_i z$ of f'',

$$(b(0, y, z)\partial_y + c(0, y, z)\partial_z)(u_i y + v_i z) \in \langle u_i y + v_i z \rangle,$$

so the map q is well defined. It follows that $D_0(C'') = \mathbb{C}[y, z](-(k - 1))$, where $k = |L \cap \mathcal{C}'| = \deg(f'')$. $\square$

Proposition IV.1. *Let $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ be a quasihomogeneous triple. Then*

$$0 \longrightarrow \mathcal{D}_0{}'(-1) \longrightarrow \mathcal{D}_0 \longrightarrow i_*\mathcal{D}_0{}''.$$

is also right exact.

Proof. It follows from Lemma IV.1 that quotienting by the Euler derivation and sheafifying yields the left exact sequence above; so it will suffice to show that $HP(D_0, t) = HP(D_0'(-1), t) + HP(i_*D_0'', t)$, where $HP(-, t)$ denotes the Hilbert polynomial. For an SRC arrangement $\mathcal{C}$ with m lines and n conics, let $d = 2n + m - 1$. We have an exact sequence:

$$0 \longrightarrow D_0(\mathcal{C}) \longrightarrow S^3 \longrightarrow S(d) \longrightarrow S(d)/J \longrightarrow 0,$$

where $S = \mathbb{K}[x, y, z]$ and J is the Jacobian ideal of the defining polynomial of $\mathcal{C}$. Since

$$HP(D_0, t) = 3\binom{t+2}{2} - \binom{t+2+d}{2} + deg(J)$$

$$HP(D_0'(-1), t) = 3\binom{t+1}{2} - \binom{t+d}{2} + deg(J'),$$

we find that

$$HP(D_0, t) - HP(D_0'(-1), t) = deg(J) - deg(J') + t - 2d + 2.$$

At each point $P_i, i = 1, \ldots, k$ on L, let n_i be the number of branches passing through P_i. By the assumption that $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ is a quasihomogeneous triple,

$$deg(J) = \sum_{p \in Sing(\mathcal{C})} \mu(p)^2 \text{ and } deg(J') = \sum_{p' \in Sing(\mathcal{C}')} \mu(p')^2.$$

Let α be the sum of Milnor numbers of points off L, so $deg(J) = \alpha + \sum_{i=1}^{k}(n_i - 1)^2$

and $deg(J') = \alpha + \sum_{i=1}^{k}(n_i - 2)^2$. Hence

$$deg(J) - deg(J') = 2\sum_{i=1}^{k}(n_i - 1) - k.$$

By Bezout's theorem,

$$\sum_{i=1}^{k}(n_i - 1) = d,$$

so $deg(J) - deg(J') = 2d - k$, and thus

$$HP(D_0, t) - HP(D_0'(-1), t) = t + 2 - k = t + 1 - (|\mathcal{C}''| - 1).$$

Since $i_*\mathcal{D}_0{}'' = \mathcal{O}_L(1 - |\mathcal{C}''|)$, this yields the result. ☐

Definition IV.5. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r$. We say $\mathcal{F}$ is $m-$regular iff $H^i\mathcal{F}(d - i) = 0$ for every $i \geq 1$. Define $reg(\mathcal{F})$ to be the least number m such that $\mathcal{F}$ is $m-$regular.

Lemma IV.2. *For a quasihomogeneous triple with* $|\mathcal{C}''| = k$*,* $reg(\mathcal{D}_0) \leq \max\{reg(\mathcal{D}_0') + 1, k - 1\}$.

Proof. Immediate from Proposition IV.1 (see [24]). ☐

1. The case where $\mathcal{C}'$ is free with exponents $\{1, k - 1, a\}$.

Suppose that $\mathcal{D}_0' = \mathcal{O}_{\mathbb{P}^2}(1 - k) \oplus \mathcal{O}_{\mathbb{P}^2}(-a)$, so for all t, $H^1(\mathcal{D}_0'(t - 1)) = 0$.

Lemma IV.3. *In this setting,* $0 \longrightarrow D_0'(-1) \longrightarrow D_0 \longrightarrow D_0'' \longrightarrow 0$ *is exact.*

Proof. The long exact sequence in cohomology arising from Proposition IV.1 and vanishing of $H^1(\mathcal{D}_0'(t))$ yield an exact sequence:

$$0 \longrightarrow \bigoplus_t H^0(\mathcal{D}_0'(-1)(t)) \longrightarrow \bigoplus_t H^0(\mathcal{D}_0(t)) \longrightarrow \bigoplus_t H^0(\mathcal{D}_0''(t)) \longrightarrow 0.$$

Theorem A.4.1 of [7] relates a graded module module to its sheaf and local cohomology (at the maximal ideal $\mathfrak{m}$) modules:

$$0 \longrightarrow H^0_{\mathfrak{m}}(D_0) \longrightarrow D_0 \longrightarrow \bigoplus_t H^0(\mathcal{D}_0(t)) \longrightarrow H^1_{\mathfrak{m}}(D_0) \longrightarrow 0.$$

This is true also for $D_0'(-1)$ and D_0''. By [7], A.4.3, $H^0_P(M) = H^1_P(M) = 0$ if $depth(M) \geq 2$. Lemma 2.1 of [16] gives the desired bound on depth for the modules of derivations, which concludes the proof. □

Proof of Theorem IV.3:

If $\mathcal{C}'$ is free with exponents $\{1, k-1, a\}$, then it follows from Lemma 2.11 that the Hilbert series of D_0 is

$$\frac{t^{a+1} + t^{k-1}}{(1-t)^3}.$$

Since $D_0' \simeq S(-k+1) \oplus S(-a)$, $reg(D_0') = \max\{k-1, a\}$. By Lemma 2.10, if $a \geq k-1$, then $reg(D_0) \leq a+1$; and if $a \leq k-2$, then $reg(D_0) \leq k$. If $a \leq k-2$, then a free resolution for D_0 is of the form :

$$0 \longleftarrow D_0 \longleftarrow S(-k+1) \oplus S(-a-1) \oplus S(-b)^d \longleftarrow S(-b)^d \longleftarrow 0.$$

For regularity reasons, b must be at most k. As this is a minimal free resolution, and it is impossible to have a syzygy on a single generator, the only situation which can actually arise occurs when $b = k$:

$$0 \longleftarrow D_0 \longleftarrow S(-k+1) \oplus S(-a-1) \oplus S(-k)^d \longleftarrow S(-k)^d \longleftarrow 0.$$

Let t_1, t_2 be two independent derivations in D_0 of degrees $deg(t_1) = a+1$ and $deg(t_2) = k-1$; our computation of the Hilbert series, combined with the fact that

$pdim(D_0) \leq 1$ means such derivations must exist. Let E, t_1', t_2' be a basis for D' with $deg(t_1') = a$ and $deg(t_2') = k-1$, and E the Euler derivation.

Now note that $t_1' \in D_0' \setminus D_0$, for otherwise in D_0 there would be an element of degree a. So $t_1'(x) \notin \langle x \rangle$. Since $D \subset D'$, then $t_1 = f_1 E + x t_1'$ and $t_2 = f_2 E + u t_2' + f t_1'$, where u is a constant, $deg(f) = k-1-a$, $deg(f_1) = a$ and $deg(f_2) = k-2$. For a resolution as above, $g t_1 = L t_2$, where L is a linear form and $deg(g) = k-(a+1)$. Hence

$$(gf_1 - Lf_2)E + (gx - Lf)t_1' + (-Lu)t_2' = 0,$$

and since E, t_1', t_2' is a basis we find that u vanishes and $gx = Lf$. But $(t_2 - f_2 E)(x) \in \langle x \rangle$ and $t_1'(x) \notin \langle x \rangle$. Since $u = 0$, x must divide f, and so $g = Lg'$ for some g'. Since $gt_1 = Lt_2$, we obtain $t_2 = g' t_1$, a contradiction.

If $a \geq k-1$, simply switch the roles of a and k above. This gives one direction of Theorem IV.3. □

2. $\mathcal{C}$ free with exponents $\{1, k-1, a+1\}$.

In order to get an appropriate vanishing, we need to dualize. Apply $Hom(-, \mathcal{O}_{\mathbb{P}^2})$ to the exact sequence

$$0 \longrightarrow \mathcal{D}_0{}'(-1) \longrightarrow \mathcal{D}_0 \longrightarrow i_* \mathcal{D}_0{}'' \longrightarrow 0.$$

The vanishing of $Hom_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^1}(t), \mathcal{O}_{\mathbb{P}^2})$ and $Ext^1_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{D}_0, \mathcal{O}_{\mathbb{P}^2})$ yield an exact sequence:

$$0 \longrightarrow \mathcal{D}_0^{\vee} \longrightarrow \mathcal{D}_0'^{\vee}(1) \longrightarrow Ext^1_S(\mathcal{O}_L(1-k), \mathcal{O}_{\mathbb{P}^2}) \longrightarrow 0.$$

The free $\mathcal{O}_{\mathbb{P}^2}$ resolution for $\mathcal{O}_L(1-k)$ is:

$$0 \longrightarrow O_{\mathbb{P}^2}(-k) \longrightarrow O_{\mathbb{P}^2}(1-k) \longrightarrow \mathcal{O}_L(1-k) \longrightarrow 0,$$

so $Ext^1_S(\mathcal{O}_L(1-k) \simeq \mathcal{O}_L(k)$.

Since $\mathcal{D}_0^\vee = \mathcal{O}_{\mathbb{P}^2}(k-1) \oplus \mathcal{O}_{\mathbb{P}^2}(a+1)$, combining this with the long exact sequence in cohomology yields a regularity bound

$$reg(\mathcal{D}_0'^\vee) \leq \max\{reg(\mathcal{D}_0^\vee) + 1, 1-k\},$$

and the exact sequence of $S-$modules:

$$0 \longrightarrow D_0^\vee(-1) \longrightarrow D_0'^\vee \longrightarrow S/L(k-1) \longrightarrow 0,$$

with $D_0^\vee = S(k-1) \oplus S(a+1)$. So:

$$HS(D_0'^\vee) = \frac{t^{-a} + t^{1-k}}{(1-t)^3}.$$

The same arguments as in the previous case show that $D_0'^\vee = S(a) \oplus S(k-1)$, hence $D(\mathcal{C}')$ is free with exponents $\{1, k-1, a\}$.

C. Addition-deletion for a conic

Let $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ be a triple of SRC arrangements in $\mathbb{P}^2$, where C is a conic in $\mathcal{C}$, and $\mathcal{C}' = \mathcal{C} \setminus C$, $\mathcal{C}'' = \mathcal{C}'|_C$. We begin with some examples.

Example IV.6. Suppose $\mathcal{C}$ is as in Example IV.4. So $D(\mathcal{C})$ is free with exponents $\{1, 2, 5\}$ and $\mathcal{C}$ is quasihomogeneous. If we delete one of the conics, the resulting arrangement $\mathcal{C}'$ is free and quasihomogeneous, with exponents $\{1, 2, 3\}$.

Theorem IV.4. *Let $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ be a quasihomogeneous triple, where $|C \cap \mathcal{C}'| = |\mathcal{C}''| =$*

k. If $k = 2m$ then the following are equivalent:

1. *$\mathcal{C}'$ is free with exponents $\{1, m, a\}$.*
2. *$\mathcal{C}$ is free with exponents $\{1, m, a+2\}$.*

When k is odd, the situation is slightly more complicated:

Example IV.7. Let $\mathcal{C}'$ be the braid arrangement A_3 defined by $V(xyz(x-z)(y-z)(x+y-z))$, and $\mathcal{C} = \mathcal{C}' \cup C$, where the conic $C = V(xy + 7xz + 13yz)$ (see Figure 9 below). $\mathcal{C}'$ is a free arrangement with exponents $\{1, 2, 3\}$, and $|\mathcal{C}''| = 7$. $\mathcal{C}$ is also

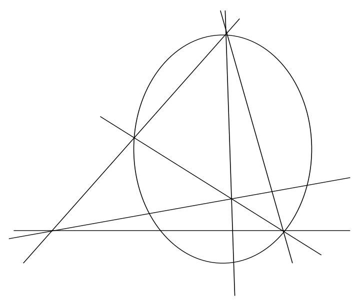

Fig. 9. Not free, quasihomogeneous SRC arrangement with 7 points on C.

quasihomogeneous, but not free.

Example IV.8. Let $\mathcal{C}$ be the quasihomogeneous SRC arrangement with defining polynomial $(x^2 - xz + 2y^2 - 2yz)xy(x+y-z)$ (Figure 10 below). $D(\mathcal{C})$ is free with exponents $\{1, 2, 2\}$. Deleting the conic yields a free line arrangement with exponents $\{1, 1, 1\}$.

Theorem IV.5. *Let $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ be a quasihomogeneous triple, where $|C \cap \mathcal{C}'| = |\mathcal{C}''| = k$. If $k = 2m+1$ then:*

1. *$exp(\mathcal{C}') = \{1, m, m\} \Longleftrightarrow exp(\mathcal{C}) = \{1, m+1, m+1\}$.*

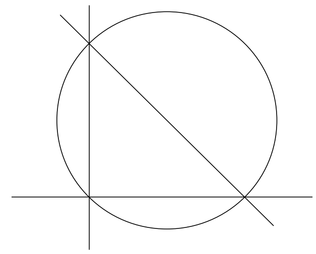

Fig. 10. Free, quasihomogeneous SRC arrangement with exponents $\{1, 2, 2\}$.

2. *if* $exp(\mathcal{C}') = \{1, m, a\}$ *with* $a \neq m$ *then* $\mathcal{C}$ *is not free.*

3. *if* $exp(\mathcal{C}) = \{1, m+1, a+1\}$ *with* $a \neq m$ *then* $\mathcal{C}'$ *is not free.*

We begin with some preliminaries. After an appropriate change of coordinates, we may suppose that $C = y^2 - xz = 0$. Let i be the composition of the maps

$$\mathbb{P}^1 \xrightarrow{v} C \hookrightarrow \mathbb{P}^2,$$

where $v(s:t) = (s^2 : st : t^2)$, and let ψ be the composite map:

$$S = \mathbb{K}[x, y, z] \xrightarrow{\phi} \mathbb{K}[s^2, st, t^2] \hookrightarrow \mathbb{K}[s, t],$$

where $\phi(x) = s^2, \phi(y) = st, \phi(z) = t^2$.

Let $\theta = a_1\partial_x + a_2\partial_y + a_3\partial_z \in D(\mathcal{C})$ be a derivation. Then $\theta(C) \in \langle C \rangle$, which means $-za_1 + 2ya_2 - xa_3 = (y^2 - xz)P$ for some $P \in S$. Via the map ψ this translates into

$$t^2\psi(a_1) - 2st\psi(a_2) + s^2\psi(a_3) = 0.$$

So there exist $Q_1, Q_2 \in \mathbb{K}[s, t]$ such that

$$\begin{aligned} \psi(a_1) &= sQ_1 \\ \psi(a_2) &= \tfrac{tQ_1 + sQ_2}{2} \\ \psi(a_3) &= tQ_2. \end{aligned}$$

If $\psi : S \longrightarrow A$ is a ring map and M is an $A-$module, let M_ψ denote the $S-$module obtained by restriction of scalars.

Proposition IV.2. *There is an exact sequence of $S-$modules*

$$0 \longrightarrow D(\mathcal{C}')(-2) \xrightarrow{\cdot C} D(\mathcal{C}) \xrightarrow{\rho} D(\mathcal{A}'')_\psi,$$

where

$$\rho(a_1\partial_x + a_2\partial_y + a_3\partial_z) = Q_1\partial_s + Q_2\partial_t,$$

for every $a_1\partial_x + a_2\partial_y + a_3\partial_z \in D(\mathcal{C})$ and Q_1, Q_2 are defined as above; and $\mathcal{A}''$ is the arrangement of the reduced points $i^{-1}(C \cap \mathcal{C}')$ in $\mathbb{P}^1$.

Proof. It is easy to check that ρ is a homomorphism. For exactness, note:

$$\begin{aligned} \theta = a_1\partial_x + a_2\partial_y + a_3\partial_z \in ker(\rho) &\leftrightarrow \quad Q_1 = 0 \text{ and } Q_2 = 0 \\ &\leftrightarrow \psi(a_1) = \psi(a_2) = \psi(a_3) = 0 \\ &\leftrightarrow \quad a_1, a_2, a_3 \in \langle y^2 - xz\rangle \\ &\leftrightarrow \quad \theta = C\theta' \text{ with } \theta' \in D(\mathcal{C}'). \end{aligned}$$

It remains to show that the image of ρ is in $D(\mathcal{A}'')$. Suppose $\alpha x + \beta y + \gamma z = 0$ is a line of $\mathcal{C}$. Let $\theta = a_1\partial_x + a_2\partial_y + a_3\partial_z \in D(\mathcal{C})$. Then

$$\alpha a_1 + \beta a_2 + \gamma a_3 = (\alpha x + \beta y + \gamma z)P_1$$

for some $P_1 \in S$. Therefore

$$\alpha\psi(a_1) + \beta\psi(a_2) + \gamma\psi(a_3) = (\alpha s^2 + \beta st + \gamma t^2)\psi(P_1),$$

which implies

$$(2\alpha s + \beta t)Q_1 + (\beta s + 2\gamma t)Q_2 = 2(\alpha s^2 + \beta st + \gamma t^2)\psi(P_1).$$

This means that $(Q_1\partial_s + Q_2\partial_t)(\alpha s^2 + \beta st + \gamma t^2) \in (\alpha s^2 + \beta st + \gamma t^2)\mathbb{K}[s,t]$. Since

$\alpha s^2+\beta st+\gamma t^2$ is the defining polynomial of the two points $i^{-1}(\{\alpha x+\beta y+\gamma z=0\}\cap C)$ in $\mathbb{P}^1$, we get that $Q_1\partial_s+Q_2\partial_t$ is a derivation on the arrangement of these 2 points.

Suppose $C'=u_0x^2+u_1xy+u_2xz+u_3y^2+u_4yz+u_5z^2=0$ is a conic in the SRC arrangement $\mathcal{C}$. Let $\theta=a_1\partial_x+a_2\partial_y+a_3\partial_z\in D(\mathcal{C})$. Computations as above show that

$$(Q_1\partial_s+Q_2\partial_t)(u_0s^4+u_1s^3t+(u_2+u_3)s^2t^2+u_4st^3+u_5t^4)$$
$$\in(u_0s^4+u_1s^3t+(u_2+u_3)s^2t^2+u_4st^3+u_5t^4)\mathbb{K}[s,t].$$

Since $u_0s^4+u_1s^3t+(u_2+u_3)s^2t^2+u_4st^3+u_5t^4$ is the defining polynomial of the 4 points $i^{-1}(C'\cap C)$ in $\mathbb{P}^1$, we get that $Q_1\partial_s+Q_2\partial_t$ is a derivation on the arrangement of these 4 points. $\square$

Let $\theta=a_1\partial_x+a_2\partial_y+a_3\partial_z\in D(\mathcal{C})_d$ such that $\rho(\theta)=s\partial_s+t\partial_t$. Then $a_1,a_2,a_3\in S_d$ with $\psi(a_1)=s^2,\psi(a_2)=st,\psi(a_3)=t^2$. Thus $d=1$ and θ is the Euler derivation in $D(\mathcal{C})$. So quotienting by the Euler derivations yields an exact sequence:

$$0\longrightarrow D_0'(-2)\xrightarrow{\cdot C}D_0\xrightarrow{\rho}(D_0(\mathcal{A}''))_\psi.$$

Since $|\mathcal{A}''|=k$, after sheafifying, $\mathcal{D}_0(\mathcal{A}'')=\mathcal{O}_{\mathbb{P}^1}(-k)$, and hence the sheafification of $D_0(\mathcal{A}'')_\psi$ is $i_*\mathcal{O}_{\mathbb{P}^1}(-k)$.

Lemma IV.4. *$HP(i_*\mathcal{O}_{\mathbb{P}^1}(-k),t)=2t+1-k.$*

Proof. CASE 1: $k=2m$. Let E be the divisor of the reduced k points $i^{-1}(C\cap\mathcal{C}')$. Then the ideal sheaf $\mathcal{I}_E=\langle f\rangle$, where $f\in\mathbb{K}[s,t]$ of degree $k=2m$. There exists $g\in S_m$, unique modulo (y^2-xz), such that $g(s^2,st,t^2)=f$. Clearly y^2-xz cannot divide g, otherwise $g(s^2,st,t^2)=0=f$, so the ideal of the reduced k points on C is $\langle y^2-xz,g\rangle$. Hence $i_*\mathcal{I}_E=\langle\bar{g}\rangle$ as an ideal of $S/\langle y^2-xz\rangle$. As an $S-$module, it has free resolution

$$0 \longrightarrow S(-2-m) \xrightarrow{\cdot C} S(-m) \longrightarrow \langle \bar{g} \rangle \longrightarrow 0,$$

which yields:

$$HP(i_*\mathcal{O}_{\mathbb{P}^1}(-2m), t) = \binom{t+2-m}{2} - \binom{t-m}{2} = 2t + 1 - 2m.$$

CASE 2: $k = 2m+1$. Let E be the divisor of the reduced k points $i^{-1}(C \cap \mathcal{C}')$. Then the ideal sheaf $\mathcal{I}_E = \langle f \rangle$, where $f \in K[s,t]$ of degree $k = 2m+1$. Let $L_1, L_2 \in \mathbb{K}[s,t]_1$ be two independent linear forms which do not divide f, and let $f_i = L_i f$. Since $\langle L_1 f, L_2 f \rangle = \langle L_1, L_2 \rangle \cap \langle f \rangle$, then $\langle f_1, f_2 \rangle$ define the same ideal sheaf on $\mathbb{P}^1$ as $\langle f \rangle$. So $\mathcal{I}_E = \langle f_1, f_2 \rangle$.

Both f_1 and f_2 are of even degree $2m+2$. So there exist $g_1, g_2 \in S = \mathbb{K}[x,y,z]$ of degree $m+1$ such that $g_i(s^2, st, t^2) = f_i, i = 1, 2$. Next we show that $J = \langle y^2 - xz, g_1, g_2 \rangle$ is the ideal of the reduced points $C \cap \mathcal{C}'$ on C. To see this, note that if $p \in C \cap \mathcal{C}'$, then $f_i(i^{-1}(p)) = 0, i = 1, 2$. So $g_i(p) = 0, i = 1, 2$, and hence $g_i \in J, i = 1, 2$. Clearly $y^2 - xz$ does not divide g_i, otherwise f_i is identically zero. Also, suppose $g_2 = \lambda g_1 + P(y^2 - xz)$, where λ is a constant. Then $f_2 = \lambda f_1$, i.e. $L_2 = \lambda L_1$; a contradiction. So J is the ideal of $2m+1$ points on the conic $y^2 - xz = 0$. By the Hilbert-Burch theorem, such an ideal is minimally generated by the 2×2 minors of

$$\begin{bmatrix} x\, y \\ y\, z \\ \alpha\, \beta \end{bmatrix}$$

where both α and β have degree m. So indeed $\langle y^2 - xz, g_1, g_2 \rangle = J$, and $i_*\mathcal{I}_E = \langle \bar{g_1}, \bar{g_2} \rangle \subseteq S/\langle y^2 - xz \rangle$. As an $S-$module it has free resolution

$$0 \longrightarrow S^2(-2-m) \xrightarrow{\left[\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}\right]} S^2(-1-m) \longrightarrow \langle \bar{g_1}, \bar{g_2} \rangle \longrightarrow 0,$$

so for the odd case we find that

$$HP(i_*\mathcal{O}_{\mathbb{P}^1}(-2m-1),t) = 2\binom{t+1-m}{2} - 2\binom{t-m}{2}$$
$$= 2t - 2m = 2t + 1 - (2m+1).$$

$\square$

Proposition IV.3. *For a quasihomogeneous triple* $(\mathcal{C}', \mathcal{C}, \mathcal{C}'' = \mathcal{C}'|_C)$, *the sequence*

$$0 \longrightarrow \mathcal{D}_0'(-2) \xrightarrow{\cdot C} \mathcal{D}_0 \longrightarrow i_*\mathcal{O}_L(-k) \longrightarrow 0$$

is exact, where $i : L \xrightarrow{[s^2:st:t^2]} \mathbb{P}^2$.

Proof. We have $HP(D_0,t) - HP(D_0'(-2),t) = 2t - 4d + 9 + (deg(J_f) - deg(J_{f'}))$, where $d+1$ is the degree of the defining polynomial f of $\mathcal{C}$ and f' is the defining polynomial of $\mathcal{C}'$. Since $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ is a quasihomogeneous triple, by Bezout's theorem $deg(J_f) - deg(J_{f'}) = 4d - 8 - k$ and hence

$$HP(D_0,t) - HP(D_0'(-2),t) = 2t + 1 - k.$$

By Lemma IV.4, this is exactly the Hilbert polynomial of the sheaf $i_*\mathcal{O}_{\mathbb{P}^1}(-k)$ associated to $D_0(\mathcal{C}'')_\psi$. $\square$

1. $\mathcal{C}'$ is free with exponents $\{1, m, a\}$.

Lemma IV.5. *For a quasihomogeneous triple such that* $\mathcal{C}'$ *is free with exponents* $\{1, m, a\}$,

$$0 \longrightarrow D_0'(-2) \longrightarrow D_0 \longrightarrow D_0(\mathcal{A}'')_\psi \longrightarrow 0$$

is exact.

Proof. As we've just seen, $\bigoplus_t H^0((i_*\mathcal{O}_L(-k))(t)) = \bigoplus_t H^0(\mathcal{O}_{\mathbb{P}^1}(2t-k)) = D_0(\mathcal{A}'')_\psi$.

With the assumption on $\mathcal{C}'$, $H^1(\mathcal{D}'_0(t-2))$ vanishes for all t, and exactness follows as in the proof of Lemma IV.3. □

It follows from the computations in the proof of Lemma IV.4 that

- If $k = 2m+1$, then $HS(D_0(\mathcal{A}'')_\psi, t) = \frac{2t^{m+1}}{(1-t)^2}$.
- If $k = 2m$, then $HS(D_0(\mathcal{A}'')_\psi, t) = \frac{t^m(1+t)}{(1-t)^2}$.

Combining these results yields the Hilbert series of D_0, and we're now ready to prove Theorems IV.4 and IV.5.

CASE 1: $k = 2m$. By Lemma IV.5,

$$HS(D_0, t) = \frac{t^m + t^{a+2}}{(1-t)^3}.$$

Since $pdim(D_0) \leq 1$, this means that there exist minimal generators $\theta, \eta \in D_0$ with $deg(\theta) = m$ and $deg(\eta) = a+2$. Suppose $\{E, \theta_1, \theta_2\}$ basis for D' with E the Euler derivation and $deg(\theta_1) = m, deg(\theta_2) = a$. We now use that $D \subset D'$.

- $m < a$. Then $\theta = fE + c\theta_1$, where c is a constant, nonzero since $\theta \in D_0$. Then $\{E, \theta, \theta_2\}$ basis for D', and then by Saito's criteria $\{E, \theta, C\theta_2\}$ is a basis for D.
- $m = a$. Then $\theta = fE + c_1\theta_1 + c_2\theta_2$, where c_1, c_2 constants, not both zero. If $c_2 \neq 0$, then $\{E, \theta_1, \theta\}$ basis for D', and then by Saito's criteria $\{E, C\theta_1, \theta\}$ is a basis for D.
- $m = a+1$. Then $\theta = fE + c_1\theta_1 + L_2\theta_2$, where c is a constant and L_2 is a linear form, not both zero. If $c_1 = 0$ then $L_2\theta_2(C) \in \langle C \rangle$. Since C is irreducible, then $\theta_2(C) \in \langle C \rangle$, and so we get $\theta_2 \in D_0$ of degree $a < m, a+2$. But this contradicts

the Hilbert series for D_0. So $c_1 \neq 0$, and so $\{E, \theta, \theta_2\}$ basis for D', and then by Saito's criteria $\{E, \theta, C\theta_2\}$ is a basis for D.

- $m = a + 2$. Then $\theta = f_1 E + c_1\theta_1 + g_1\theta_2$, where c_1 is a constant and g_1 is a quadratic form, not both zero and $\eta = f_2 E + c_2\theta_1 + g_2\theta_2$, where c_2 is a constant and g_2 is a quadratic form, not both zero. If $c_1 = c_2 = 0$, then either $g_i = c_i' C, c_i' \neq 0, i = 1, 2$ or $\theta_2 \in D_0$ (and this is a contradiction, because $deg(\theta_2) = a$). Therefore we get $c_2'\theta - c_1'\eta \in D_0 \cap ES = \{0\}$. Contradiction with the minimality of $\theta, \eta \in D_0$. So, say $c_2 \neq 0$, then $\{E, \eta, \theta_2\}$ basis for D', and then by Saito's criteria $\{E, \eta, C\theta_2\}$ is a basis for D.

- $m > a + 2$. Then $\theta = f_1 E + c_1\theta_1 + g_1\theta_2$, where c_1 is a constant and g_1 is a polynomial, not both zero and $\eta = f_2 E + c_2\theta_1 + g_2\theta_2$, where c_2 is a constant and g_2 is a quadratic form, not both zero. If $c_1 = c_2 = 0$, then $g_1 = Cg_1', g_1' \neq 0$ and $g_2 = c_2' C$, c_2' nonzero constant. Same trick as before gives us the contradiction. So $c_1 \neq 0$ or $c_2 \neq 0$. Again, use Saito's criteria to get the desired result.

CASE 2: $k = 2m + 1, m = a$. By Lemma IV.5,

$$HS(D_0, t) = \frac{2t^{m+1}}{(1-t)^3}.$$

As in the previous case, this means that there exists minimal generators elements $\theta, \mu \in D_0$ with $deg(\theta) = deg(\mu) = m+1$. Suppose $\{E, \theta_1, \theta_2\}$ is a basis for D' where E is the Euler derivation and $deg(\theta_1) = m, deg(\theta_2) = m$. So $\theta = f_1 E + L_1\theta_1 + K_1\theta_2$ and $\mu = f_2 E + L_2\theta_1 + K_2\theta_2$, where L_1, L_2, K_1, K_2 are linear forms, and for any $i = 1, 2$, L_i, K_i cannot be simultaneously zero.

We get that $L_2\theta - L_1\mu - (L_2 f_1 - L_1 f_2)E = (L_2K_1 - L_1K_2)\theta_2$ is an element in $D(\mathcal{C})$.

But θ_2 is in $D(\mathcal{C}')$ and $\theta_2(C) \notin \langle C \rangle$, otherwise we would have a degree m element in D_0, which is inconsistent with the Hilbert Series. Hence $L_2K_1 - L_1K_2 = cC$, where c is a constant.

If $c = 0$, then $L_1 = uK_1, L_2 = uK_2, u \neq 0$ or $L_1 = vL_2, K_1 = vK_2, v \neq 0$, where u, v are constants. We also get that $K_2f_1 = K_1f_2$ and $L_2f_1 = L_1f_2$. If $L_1 = uK_1, L_2 = uK_2, u \neq 0$, and $K_1 \neq ct \cdot K_2$ we get $\theta = K_1(gE + u\theta_1 + \theta_2)$. Since $K_1 \neq 0$ (otherwise $L_1 = 0$) then, as $\theta(C) \in \langle C \rangle$, we get $(gE + u\theta_1 + \theta_2)(C) \in \langle C \rangle$, and therefore we get a degree m derivation in $D(\mathcal{C})$, a contradiction. If $K_1 = ct \cdot K_2$, then we obtain θ and μ not minimal generators. If $c \neq 0$, then we find $\det[E, \theta, \mu] = cC \det[E, \theta_1, \theta_2]$, and Saito's criteria shows that $\{E, \theta, \mu\}$ is a basis for $D(\mathcal{C})$.

2. $\mathcal{C}$ is free

As in the previous section, apply $Hom(-, \mathcal{O}_{\mathbb{P}^2})$ to the exact sequence

$$0 \longrightarrow \mathcal{D}_0'(-2) \xrightarrow{\cdot C} \mathcal{D}_0 \longrightarrow i_*\mathcal{O}_L(-k) \longrightarrow 0.$$

Since $i_*\mathcal{O}_L(-k)$ is supported on the conic C, $Hom(i_*\mathcal{O}_L(-k), \mathcal{O}_{\mathbb{P}^2}) = 0$. The assumption that D_0 is free implies that $Ext^1_S(\mathcal{D}_0, \mathcal{O}_{\mathbb{P}^2}) = 0$. This yields an exact sequence:

$$0 \longrightarrow \mathcal{D}_0^\vee \longrightarrow \mathcal{D}_0'^\vee(2) \longrightarrow Ext^1_S(i_*\mathcal{O}_L(-k), \mathcal{O}_{\mathbb{P}^2}) \longrightarrow 0.$$

As D_0 free with known exponents, so also is $D_0^\vee$, and the Hilbert Series is known. The proof of Lemma IV.4 provides a free resolution of $i_*\mathcal{I}_E$, which allows us to compute $Ext^1_S(i_*\mathcal{I}_E, S)$. Combining everything yields the Hilbert Series of $D_0'^\vee$, and the result follows as in the previous analysis.

D. Freeness of SRC arrangements is not combinatorial

Let $\mathcal{A}$ be the union of the five smooth conics:

$$
\begin{aligned}
C_1 &= (x-3z)^2+(y-4z)^2-25z^2=0\\
C_2 &= (x-4z)^2+(y-3z)^2-25z^2=0\\
C_3 &= (x+3z)^2+(y-4z)^2-25z^2=0\\
C_4 &= (x+4z)^2+(y-3z)^2-25z^2=0\\
C_5 &= \quad (x-5z)^2+y^2-25z^2=0
\end{aligned}
$$

$\mathcal{A}$ has 13 singular points, all ordinary. At 10 of these points only two branches of $\mathcal{A}$ meet, while at the points $(0:0:1),(1:i:0),(1:-i:0)$, all five conics meet. The Milnor and Tjurina numbers agree at all singularities except $(0:0:1)$, where $\tau = 15$ and $\mu = 16$. Adding lines L_1, L_2, L_3 connecting $(0:0:1),(1:i:0),(1:-i:0)$ yields a free SRC arrangement $\mathcal{C}$, with $D_0(\mathcal{C}) = S(-6)^2$.

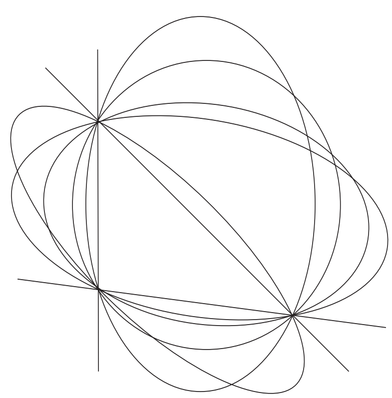

Fig. 11. Terao's conjecture fails for SRC arrangements with this real picture.

Next, let $\mathcal{A}'$ be the union of the following five smooth conics:

$$\begin{aligned}
C_1 &= \ x^2 + 8y^2 + 21xy - xz - 8yz = 0 \\
C_2 &= \ x^2 + 5y^2 + 13xy - xz - 5yz = 0 \\
C_3 &= \ x^2 + 9y^2 - 4xy - xz - 9yz = 0 \\
C_4 &= \ x^2 + 11y^2 + xy - xz - 11yz = 0 \\
C_5 &= x^2 + 17y^2 - 5xy - xz - 17yz = 0
\end{aligned}$$

$\mathcal{A}'$ is combinatorially identical to $\mathcal{A}$, but at the points $(0:0:1), (1:0:1), (0:1:1)$ where all the branches meet, $\tau = 15$ and $\mu = 16$.

Adding the lines connecting these 3 points yields an SRC arrangement $\mathcal{C}'$ which is combinatorially identical to $\mathcal{C}$ but *not free* (see Figure 11 for the real picture); the free resolution of $D_0(\mathcal{C}')$ is:

$$0 \longrightarrow S(-8)^2 \longrightarrow S(-7)^4 \longrightarrow D_0(\mathcal{C}') \longrightarrow 0.$$

CHAPTER V

GORENSTEIN EVALUATION CODES

A. Introduction

Definition V.1. A *linear code C of dimension k, length n on* $\mathbb{F}_q$ is the image of a linear map of rank k,

$$\mathbb{F}_q^k \longrightarrow \mathbb{F}_q^n.$$

The elements of C are called *codewords*. $\mathbb{F}_q$ is the finite field with $q = p^s$ elements.

Definition V.2. For $x, y \in \mathbb{F}_q^n$, the *Hamming distance* is

$$d(x, y) = |\{i, 1 \leq i \leq n : x_i \neq y_i\}|.$$

$d(x, y)$ is the number of positions where the components of the two vectors x and y differ.

Definition V.3. *The minimum distance* of a code C is

$$d = \min\{d(x, y) : x \neq y \in C\}.$$

Codes are used in communication of information, to add some redundancy to allow correction of errors arising during signal transmission. $[n, k, d]$ are the parameters of a code; good codes are ones for which $\frac{k}{n}$ is not too small, but for which d is relatively large. Finding such codes is one of the main problems in coding theory. So for fixed k and n, finding d, or at least bounds of it, is important. *The Singleton bound* says that for any linear code

$$d \leq n - k + 1.$$

In this chapter we will discuss lower bounds for the minimum distance.

Example V.1. One of the codes used a lot in applications (e.g., in Compact Disc audio system) are *Reed-Solomon codes*: Let $\mathbb{F}_q$ be a finite field and let α be a primitive element of $\mathbb{F}_q$ (i.e., α generates the multiplicative cyclic group $\mathbb{F}_q \setminus \{0\}$). Let $n = q-1$ and fix $k < q$. Consider $L_{k-1} = \{\sum_{i=0}^{k-1} a_i t^i : a_i \in \mathbb{F}_q\}$ the vector space of polynomials of degree at most $k-1$.

The Reed-Solomon code is given by:

$$RS(k,q) = \{(f(1), f(\alpha), \ldots, f(\alpha^{q-2})) \in \mathbb{F}_q^{q-1} : f \in L_{k-1}\}.$$

This is a code of dimension k, block length $q-1$ and minimum distance $d = q-k$. Note that the Singleton bound is attained.

The linear codes in Example V.1 are a particular case of *evaluation codes*. The name arises because code words are obtained by evaluating the polynomials f. Let V be a variety in $\mathbb{P}^m$ defined over the finite field $\mathbb{F}_q$, with $\Gamma = \{p_1, \ldots, p_n\}$ a set of rational points on V. Let $R = \mathbb{F}_q[x_0, \ldots, x_m]$ and let R_a denote the vector space of homogeneous polynomials of degree a. Choose $f_0 \in R_a$ such that $f_0(p_i) \neq 0, \forall i \in \{1, \ldots, n\}$. Define a linear map:

$$e_a(\Gamma) : R_a \longrightarrow \mathbb{F}_q^n$$

$$f \mapsto (\frac{f(p_1)}{f_0(p_1)}, \ldots, \frac{f(p_n)}{f_0(p_n)}).$$

The image of $e_a(\Gamma)$ is a linear code of block length n, denoted $C(\Gamma)_a$ and called *the evaluation code associated to* Γ.

A simple computation shows that the dimension of $C(\Gamma)_a$ is

$$k_a = HF(R/I_\Gamma, a),$$

where I_Γ is the ideal of the points Γ. In [11], the authors study the minimum distance of $C(\Gamma)_a$ in terms of the properties of the ideal I_Γ, when Γ is a zero-dimensional a complete intersection. In the next section we will do the same, but for the more general case, when I_Γ is a Gorenstein ideal.

B. Gorenstein evaluation codes

Definition V.4. Let $R = \mathbb{K}[x_0, \dots, x_n]$ and let I be a homogeneous ideal. Let $A = R/I$. We say that A (or I) is *Artinian* iff there exists k such that $A_k = 0$. If s is such that $A_{s+1} = 0$ and $A_s \neq 0$, then s is called *the socle degree* of A.

Let I be a homogeneous ideal in R. By *Artinian reduction*, we mean an ideal $J = I + \langle L_1, \dots, L_m \rangle$ such that R/J is an Artinian ring, where L_i are generic linear forms: L_i is a nonzero divisor in $R/\langle I, L_1, \dots, L_{i-1} \rangle$. For example, if I is the ideal of a set of points in $\mathbb{P}^n$, then $m = 1$ and $V(L_1)$ is any hyperplane missing all the points.

Definition V.5. An ideal I is *Gorenstein*, if s is the socle degree of the Artinian reduction $A = R/J$, and $\dim_{\mathbb{K}} A_s = 1$.

Example V.2. if $\Gamma \subset \mathbb{P}^m$ is a complete intersection of hypersurfaces of degrees $d_1, d_2, \dots, d_m$, then Γ is Gorenstein of socle degree $l = (\sum_{i=1}^m d_i) - m$ ([8], Theorem CB8).

The Hilbert function of an Artinian Gorenstein ideal is symmetric:

Proposition V.1. *If R/I is Gorenstein of socle degree s, then*

$$HF(R/I, i) = HF(R/I, s - i), \forall i \in \{0, \dots, s\}.$$

Let $\Gamma \subset \mathbb{P}^m$ be a nondegenerate (i.e. not contained in a hyperplane) finite set of n points, and, for a positive integer a, let $C(\Gamma)_a$ be the linear code associated to Γ.

As noticed in [12], Proposition 6, the minimum distance for $C(\Gamma)_a$ is

$$d_a = n - \max_{\Gamma' \subset \Gamma}\{|\Gamma'| : \dim(I_{\Gamma'})_a > \dim(I_\Gamma)_a\},$$

where I_Γ is the ideal of Γ.

Let $G_a = \{\Gamma' \subset \Gamma : \dim(I_{\Gamma'})_a > \dim(I_\Gamma)_a\}$. Notice that $\Gamma \notin G_a$ and we must be aware that two maximal elements in G_a (under $\subseteq$) might have different sizes.

Lemma V.1. *$\forall a \geq 1$ we have $d_{a-1} \geq d_a$.*

Proof. Let $\tilde{\Gamma} \in G_{a-1}$ such that $d_{a-1} = n - |\tilde{\Gamma}|$. So $\dim(I_{\tilde{\Gamma}})_{a-1} > \dim(I_\Gamma)_{a-1}$. Therefore $\exists f \in (I_{\tilde{\Gamma}})_{a-1}$ and $\exists p \in \Gamma - \tilde{\Gamma}$ such that $f(p) \neq 0$.

Let L be a linear form not vanishing at p. We have $Lf \in (I_{\tilde{\Gamma}})_a$ and $L(p)f(p) \neq 0$. This means that $Lf \notin (I_\Gamma)_a$, and hence $\dim(I_{\tilde{\Gamma}})_a > \dim(I_\Gamma)_a$. So, $\tilde{\Gamma} \in G_a$. Therefore $n - d_a \geq |\tilde{\Gamma}| = n - d_{a-1}$ and hence the result. □

Lemma V.2. *Let Γ' be a maximal element in G_a and suppose $|\Gamma - \Gamma'| \geq 2$. Then $\dim(I_{\Gamma'})_{a-1} = \dim(I_\Gamma)_{a-1}$.*

Proof. If $\dim(I_{\Gamma'})_{a-1} > \dim(I_\Gamma)_{a-1}$, then $\exists f \in (I_{\Gamma'})_{a-1}$ and $\exists p \in \Gamma - \Gamma'$ such that $f(p) \neq 0$. Let $q \in \Gamma - (\Gamma' \cup \{p\})$ and let $\Gamma'' = \Gamma' \cup \{q\}$.

Let L linear form such that $L(q) = 0$ and $L(r) \neq 0, \forall r \in \Gamma - \Gamma'$. So $Lf \in (I_{\Gamma''})_a$ and $L(p)f(p) \neq 0$. So $\Gamma'' \in G_a$ and $\Gamma'' \supsetneq \Gamma'$. This contradicts the maximality of Γ'. So $\dim(I_{\Gamma'})_{a-1} = \dim(I_\Gamma)_{a-1}$. □

Proposition V.2. *If $\exists s$ such that $d_s \geq 2$, then $\forall a \in \{1, \ldots, s\}$, we have $d_{a-1} - 1 \geq d_a$.*

Proof. Suppose $d_a = d_{a-1}$ for some $a \in \{1, \ldots, s\}$.

Let $\Gamma' \in G_{a-1}$ such that $d_{a-1} = n - |\Gamma'|$. In proof of Lemma V.1 we saw that $\Gamma' \in G_a$. Since $d_a = d_{a-1}$, then Γ' is a maximal element in G_a. If $|\Gamma - \Gamma'| \geq 2$, then by Lemma V.2, $\dim(I_{\Gamma'})_{a-1} = \dim(I_\Gamma)_{a-1}$. But this contradicts the fact that $\Gamma' \in G_{a-1}$.

So $|\Gamma'| = |\Gamma| - 1$. $d_a = d_{a-1} = n - |\Gamma'| = n - (n-1) = 1$. But this is a contradiction since, by Lemma V.1, $d_a \geq d_s \geq 2$. Therefore, $\forall a \in \{1, \dots, s\}$, we have $d_{a-1} - 1 \geq d_a$. □

Corollary V.1. *If $\exists s$ such that $d_s \geq 2$, then $\forall a \in \{1, \dots, s\}$ we have $d_a \geq s - a + 2$.*

Therefore Theorem 3.2 in [11] generalizes to the Gorenstein case as follows:

Proposition V.3. *Let $\Gamma \subset \mathbb{P}^m$ be a reduced nondegenerate Gorenstein zero-dimensional scheme of socle degree l. For any $1 \leq a \leq l-1$, the evaluation code $C(\Gamma)_a$ has minimum distance $d_a \geq l - a + 1$.*

Proof. We will apply Corollary V.1 for the case $s = l - 1$. So we need to prove that $d_{l-1} \geq 2$. Basically, the proof of this is outlined in [8]. Suppose $d_{l-1} = 1$. Then there exists Γ' such that $\Gamma = \Gamma' \cup \{p\}$ and $f \in (I_{\Gamma'})_{l-1}$ with $f(p) \neq 0$. From now on let $I = I_\Gamma$ and $I' = I_{\Gamma'}$.

Let L be a nonzerodivisor on R/I such that $A = R/\langle I, L\rangle$ is an Artinian Gorenstein ring of socle degree l. If L is zerodivisor in R/I' then $\exists h \notin I'$ such that $Lh \in I'$. So $(Lh)(q) = 0 \forall q \in \Gamma'$ and $\exists q_0 \in \Gamma'$ such that $h(q_0) \neq 0$. Let L' be a linear form such that $L'(p) = 0$. Then $L(L'h)(r) = 0, \forall r \in \Gamma$. Since L is not a zerodivisor, then $(L'h)(r) = 0, \forall r \in \Gamma$. So $L'(q_0) = 0$. So every linear form vanishing at p should vanish at q_0. So $p = q_0 \in \Gamma'$. Contradiction. So L is a nonzerodivisor in R/I'. Therefore we get two exact sequences:

$$0 \longrightarrow R/I(-1) \longrightarrow R/I \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow R/I'(-1) \longrightarrow R/I' \longrightarrow A' \longrightarrow 0,$$

where $A' = R/\langle I', L\rangle$.

We find that the Hilbert functions of A and A' satisfy:

$$h_A(l-1) = h_{R/I}(l-1) - h_{R/I}(l-2)$$

and

$$h_{A'}(l-1) = h_{R/I'}(l-1) - h_{R/I'}(l-2).$$

Hence, $\dim(I'_{l-1}) - \dim(I_{l-1}) = \sum_{j=0}^{l-1}(h_A(j) - h_{A'}(j))$. If $\bar{I}'$ is the image of I' in A, then from the exact sequence $0 \longrightarrow \bar{I}' \longrightarrow A \longrightarrow A/\bar{I}' \cong A' \longrightarrow 0$, we get $\dim(I')_{l-1} - \dim I_{l-1} = \sum_{j=0}^{l-1} \dim(\bar{I}'_j)$.

We have that $\{p\}$ is the subscheme residual to Γ'. Let $\bar{m_p}$ be the image of the ideal of p in A. Since A is Artinian Gorenstein of socle degree l, we have a pairing:

$$A_j \times A_{l-j} \longrightarrow A_l \cong \mathbb{K}.$$

Since $I : I' = m_p$, we have $\bar{I}'_j = ((\bar{m_p})_{l-j})^{\perp}$ and hence $\dim(\bar{I}'_j) = \dim(A_{l-j}) - \dim((\bar{m_p})_{l-j})$.

Summing these, we get $\dim(I')_{l-1} - \dim I_{l-1} = \sum_{j=0}^{l-1} h_{A/\bar{m_p}}(l-j) = \sum_{j=0}^{l-1} h_{R/\langle m_p, L\rangle}(l-j)$, since $A/\bar{m_p} \cong R/\langle I, m_p, L\rangle$ and because $I \subset m_p$, as $p \in \Gamma$.

We supposed that $\exists f \in (I')_{l-1}$ such that $f(p) \neq 0$. If $L(p) = 0$, then $Lf \in I$ and hence L is a zerodivisor in R/I. Contradiction with the way we picked L. So $L(p) \neq 0$ and we have an exact sequence:

$$0 \longrightarrow R/m_p(-1) \longrightarrow R/m_p \longrightarrow R/\langle m_p, L\rangle \longrightarrow 0$$

which gives us $h_{R/\langle m_p, L\rangle}(l-j) = h_{R/m_p}(l-j) - h_{R/m_p}(l-j-1)$.

Therefore, $\dim(I')_{l-1} - \dim I_{l-1} = h_{R/m_p}(l) - h_{R/m_p}(0) = 1 - 1 = 0$. So $(I_{\Gamma'})_{l-1} = (I_\Gamma)_{l-1}$ and hence $d_{l-1} \geq 2$. $\square$

C. Applications

We conclude with some applications of the results in the previous section to the case of points in $\mathbb{P}^2$. In the first example we will use the following result:

Proposition V.4. *([8], Proposition 1) Let a be a positive integer and let $\Gamma = \{p_1, \ldots, p_n\} \subset \mathbb{P}^2$ be any collection of $n \leq 2a+2$ distinct points. The points of Γ fail to impose independent conditions on curves of degree a (i.e. $|\Gamma| > h_\Gamma(a)$)) iff either $a+2$ of the points of Γ are collinear or $n = 2a+2$ and Γ is contained in a conic.*

Using this proposition we will estimate the minimal distance d_a for the evaluation code $C(\Gamma)_a$. Suppose that not all the points of Γ are on a line in $\mathbb{P}^2$ and $|\Gamma| \leq 2a+1$.

Γ imposes independent conditions on forms of degree a iff $rank(e_a) = \dim C(\Gamma)_a = n$. So, from the Singleton bound, this tells us that $d_a = 1$.

If Γ fails to imposes independent conditions on forms of degree a, then by the result above $a+2$ points of Γ are on a line.

Since Γ is not all on a line, then $\exists \Gamma' \subset \Gamma, |\Gamma'| = a+2$ such that $\dim(I_{\Gamma'})_1 > \dim(I_\Gamma)_1 = 0$. So $n - d_1 \geq a+2$. By Lemma V.1, $d_a \leq d_1 \leq n-a-2$. If even more happens, that is $d_a \geq 2$, then by Proposition V.2 $d_a \leq d_1 - (a-1) \leq n - 2a - 1$. But $d_a \geq 2$, so we get $n \geq 2a+3$. But this contradicts the assumption we made: $n \leq 2a+1$. So $d_a \leq 1$.

We proved the following:

Corollary V.2. *Let a be an integer and let $\Gamma \subset \mathbb{P}^2$ be a collection of $a+3 \leq n \leq 2a+1$ distinct points not all on a line. Then $d_a = 1$.*

Example V.3. Now, let's look at some Gorenstein not complete intersection zero-dimensional reduced schemes.

In general, let I be a homogeneous ideal in $R = \mathbb{K}[x_0, \ldots, x_m]$. By definition, R/I is Gorenstein ring iff R/I has a graded minimal free resolution of the form

$$0 \longrightarrow F_k \longrightarrow F_{k-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

where $k = codim(I)$ and $F_k \cong R(-\alpha)$.

In the case of Gorenstein zero-dimensional subscheme Γ in $\mathbb{P}^m$ of socle degree l, the minimal free resolution for R/I_Γ will have $k = m$ and $F_m \cong R(-(l+m))$.

By [15], if $codim(I) = 2$, then every Gorenstein ring R/I is a complete intersection. So, to find Gorenstein not complete intersection zero-dimensional reduced schemes we must look at sets of points in $\mathbb{P}^3$. The first such example is a set Γ of 5 points in general position in $\mathbb{P}^3$. The minimal free resolution for R/I_Γ is

$$0 \longrightarrow R(-5) \longrightarrow R^5(-3) \longrightarrow R^5(-2) \longrightarrow R \longrightarrow R/I_\Gamma \longrightarrow 0.$$

So Γ is Gorenstein of socle degree 2 and not a complete intersection (I_Γ is minimally generated by 5 elements and not just 3).

From Proposition V.2, we get that $d_1 \geq 2 - 1 + 1 = 2$. From Singleton bound we also have $d_1 \leq 5 - rank(\phi_1) + 1$. But $rank(\phi_1) = \dim(R_1) = 4$ since the 5 points do not lie in a plane in $\mathbb{P}^3$. So we get $d_1 = 2$.

We can generalize this result to the following:

Proposition V.5. *Let $\Gamma = \{p_0, \ldots, p_{m+1}\}$ be a set of $m+2$ points in general position in $\mathbb{P}^m$. Then we have:*

$$d_1 = 2, d_i = 1, \forall i = 2, \ldots m + 1.$$

Proof. By [14], Exercise 1.6., modulo a change of coordinates we can suppose $p_0 = (1 : 0 \cdots : 0), \ldots, p_m = (0 : \cdots : 0 : 1), p_{m+1} = (1 : 1 : \cdots : 1)$.

Since Γ is in general position, no $m+1$ points of Γ are on a hyperplane in $\mathbb{P}^m$. But observe that $p_0, \ldots p_{m-1} \in ker(x_m)$. So, by definition $d_1 = m+2-m = 2$.

For $m \geq i \geq 1$ consider the hypersurface C of degree $i+1$ given by the equation $x_0 x_1 \cdots x_i = 0$. Observe that all the points of Γ, but p_{m+1} are on C. So $d_{i+1} = m+2-(m+1) = 1$. $\square$

We will end, by mentioning that we could have proven $d_1 = 2$ using the same argument as for the case of 5 general points in $\mathbb{P}^3$. By [9], Theorem 4.3. ($\mathbb{K}$ algebraically closed), a set of general $m+2$ points in $\mathbb{P}^m$ is Gorenstein, and hence with the Singleton bound and our Proposition V.2, we get the desired result.

Example V.4. This next example will show that the minimal distance can not be determined just from the Hilbert function.

Let's consider the following sets of 6 points each in $\mathbb{P}^2$, both complete intersections of a conic C and a cubic Q. For the first set Γ_1, suppose C is an irreducible conic and Q is a union of 3 lines. For the second set Γ_2, take C to be reducible (union of 2 lines) and Q same as before. It is easy to check that $d_1(\Gamma_1) = 6-2 = 4$ and $d_1(\Gamma_2) = 6-3 = 3$, though both R/I_{Γ_1} and R/I_{Γ_2} have the same free resolution:

$$0 \longrightarrow R(-5) \longrightarrow R(-3) \oplus R(-2) \longrightarrow R \longrightarrow R/I_{\Gamma_i} \longrightarrow 0,$$

and hence the same Hilbert function.

The minimum distance d_1 for both cases verifies the lower bound in [11], Theorem 3.2: $d_1 \geq (5-2-1)-1+2 = 3$.

In [2], Theorem 1, the authors give the following lower bound for the minimum distance of an evaluation code: Let X be a linear general position (i.e. every $n+1$ points of X span $\mathbb{P}^n$) reduced complete intersection zero-dimensional in $\mathbb{P}^m$. Then, for $1 \leq a \leq s$, $d_a \geq m(s-a)+2$, where $s = (\sum_{i=1}^{m} d_i) - m - 1$ and d_i are the degrees of the hypersurfaces defining X. Observe that Γ_1 is in the situation described above. We get $d_1(\Gamma_1) \geq 2(2-1)+2=4$. So the inequality obtained in [2] is sharp.

CHAPTER VI

SUMMARY AND CONCLUSIONS

Throughout this dissertation we saw how Hilbert function computations and homology can answer various questions, from splines approximation and hyperplane arrangements to algebraic coding theory. There are interesting questions that arise just from solving the four problems in the dissertation. The author will address these in the near future.

In Chapter II, we show that a certain conjecture about the dimension of the space of piecewise polynomial functions which are C^r on a planar triangulation is tight. The proof relies on the explicit computation of the nonvanishing of the first local cohomology module described in [28]. Basically we showed that $HF(N, 2r) \neq 0$, for a specific graded module N. A natural question will be to check the validity of the conjecture for this particular example. That is, to show that $HF(N, 2r + 1) = 0$ for the same module N and for any r. Plus, it is also natural to think of the algebra structure of these space, in the view of [25] and [3] as subalgebras of Stanley-Reisner ring (a very nice intuitive introduction to these rings can be found in [23]).

In Chapter III, we studied the problem of $k-$ formality of hyperplane arrangements and we proved a criterion about the $k-$formality of graphic arrangements, that relates it to the vanishing of the homology of flag complexes for graphs. Yuzvinsky ([32]) gives two examples of line arrangements, one formal, the other not, but with the same combinatorics. A different approach to study formality will be to study the algebras introduced by Orlik and Terao in [18]. Together with H. Schenck, after a small modification of these algebra, we were able to see changes in the free resolutions of the algebras associated to the two examples above.

In Chapter IV we study extensively the problem of freeness of configurations of smooth rational curves in $\mathbb{P}^2$. Since these configurations look similar to line arrangements in $\mathbb{P}^2$, we generalized the notion of freeness in the same spirit Saito ([20]) and Terao ([29]) did. Using sheaf theory, Hilbert function computations, free resolutions we prove an addition-deletion type theorem and we give a counterexample to Terao's conjecture in this new setup. The addition-deletion theorem requires very nice behavior of the singularities (Tjurina and Milnor numbers must be equal). So one subject of research in the future will be to find a good criteria of quasihomogeneity (when do these two numbers are equal?) of line-conic configurations.

In Chapter V we approach a very important problem in algebraic coding theory: finding the minimal distance for linear codes. For evaluation codes on zero-dimensional schemes, the problem translates into a question in computational commutative algebra ([13] and [11]). In [11], the authors find a lower bound for the minimal distance of complete intersection evaluation codes. In this chapter we generalize the result to Gorenstein evaluation codes and we express the minimal distance bound in terms of the socle degree. A very nice survey on Gorenstein rings is [15]. A natural question is to find the minimal distance of Cohen-Macaulay evaluation codes in terms of the regularity of the zero-dimensional scheme.

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